# HINTS,

## THEORETICAL, ELUCIDATORY, AND PRACTICAL,

FOR THE USE OF

## TEACHERS

OF

# ELEMENTARY MATHEMATICS,

AND OF

## SELF-TAUGHT STUDENTS;

WITH ESPECIAL REFERENCE TO

VOL. I. OF HUTTON'S COURSE, AND SIMSON'S EUCLID,

AS TEXT-BOOKS:

ALSO,

A SELECTION OF MISCELLANEOUS TABLES, AND AN APPENDIX ON THE GEOMETRICAL DIVISION OF PLANE SURFACES.

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BY

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# PREFACE.

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In the year 1837, some recent extensions of the qualifications for admission into the Royal Military Academy, the completion of the eleventh edition of the Course of Mathematics, and some other circumstances which need not here be detailed, rendered necessary for circulation among the Junior Masters of the Royal Military Academy, an official Paper of Directions, to produce, as far as practicable, uniformity of system in the lower departments of the Institution. This was accordingly prepared.

It occurred to me, some time after my retirement from my official engagements, that these

directions, with additions and enlargements, might be of utility to self-taught students, carrying on mathematical or other scientific pursuits, as well as to those valuable members of the community whose talents and industry are employed in the earlier branches of elementary instruction.

In sketching the first openings of mathematical science, and the value of mathematical reasonings, I have here aimed to show their utility, not only, as others have done, on account of the positive knowledge which may be gathered on the various topics that this department of science exhibits, but also because of the collateral effects which grow out of the study, as a mental discipline. I have endeavoured to show that mathematical principles must be deposited in the mind, as far as possible, by a classified and logical grouping, with a prospective reference, which the laws of memory and association foster and strengthen, and with an attention stimulated and kept alive by a sense of their value, as they occur as parts of a whole. I have also aimed to show that arithmetic, algebra, and geometry, while in some respects they differ,

in others coalesce; that while in several particulars they may be pursued simultaneously, they often must proceed independently, each calling forth its own resources.

In referring to R. Simson's Euclid, (as the edition most usually adopted,) I have pointed out more of his errors and defects than may perhaps be anticipated by some persons; because, though his is a very admirable edition, it is by no means faultless; and he often labours under the strange hypothesis that Euclid himself is more correct and scientific than any of his annotators, many of whom were very extraordinary geometricians. I have, therefore, with a view to the ultimate accuracy of others, pointed out several of his mistakes; a course which I persuade myself will be approved by the reader, whether learner or preceptor, who leans less to authority than to evidence.

The Tables in the small collection given after the summary of Hints, will each, I trust, mark its practical character, and evince its value, especially to young mathematicians; either in shortening computations, in supplying praxes, in facilitating

investigations, in announcing useful results, or in developing the progress of methods, and the mechanism of certain transformations; and severally help to show that the best motive to study springs from a clear perception of the value of the thing to be learned.

With the same views, also, I have introduced the Appendix prepared and published about thirty years since, on the Geometrical Division of Surfaces, to which I have now added some useful propositions from the geometry of Hirsch, and a few valuable results from Euler.

It has ever been with me a principle, that even rudimental teaching must be thorough, and must be founded upon the essential character of the mind, as accretive; and that when once it is rightly proceeded upon, the teaching of any science will not remain quiescent, but will advance to completion and success. This result I have had mainly in view on the present occasion.

OLINTHUS GREGORY.

*Woolwich Common, June* 9, 1840.

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## ERRATA.

Page 45, line 9, *for* $a^n + b$, *read* $a^n + b^n$.

—, — 14, *for* $a + a^n$, *read* $a^m + a^n$.

129, — 15, *for* $\frac{1}{5^1}$, *read* $\frac{1}{5^{\frac{1}{2}}}$.

138, — 8 from bottom, *for* Herschell's, *read* Herschel's.

148, — 17, *for* $x^3 = 13x^2$, *read* $x^3 - 13x^2$.

# HINTS.

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*Summary of hints and directions, for conducting the mathematical instruction, after the first admissions of Gentlemen Cadets into the Royal Military Academy.*

THE recent extension of the qualifications for admission into this Institution, the completion of a new edition of the course of mathematics, and some other circumstances of moment, which need not here be detailed, being regarded as rendering necessary the preparation of a paper of hints and directions, with preliminary and occasional elucidatory remarks, for the use of the masters in the lower sections of the Royal Military Academy, (especially with the view of enforcing an approximation to uniformity of system,) it is here presented, in obedience to the Master-General's directions.

1. Let it be always borne in mind, that *education* consists in so cultivating the intellect, as to render it a more powerful and more exact instrument than it could otherwise be, for the acquisition, the propagation, and the discovery of truth; and at the same time, a more certain guide for the regulation of conduct, and the successful pursuit of professional objects. In other words, education involves the adequate preparation of youth for the business of mature life.

2. The faculty of reasoning, so far as it can be disciplined (and that is very far,) by a judicious mixture of theoretical and practical teaching, receives such a discipline from mathematical study.

3. Others, besides mathematicians, have regarded geometry especially as an excellent logic; and the reason may be at once assigned:—for, when the definitions employed in the inquiry are clear, when the postulates cannot be refused, nor the axioms denied, or hesitated upon; when, from the distinct contemplation and comparison of figures, their properties are derived by means of a perpetual, well-conducted train of indubitable consequences, the objects being all along kept in view, and the *attention ever fixed upon them*, there is naturally acquired a habit of reasoning, close, exact, and methodical; a habit which strengthens and sharpens the mind; and, being transferred to

other subjects, is of general use in the inquiry after truth.

4. The same advantages, in a greater or less degree, flow from every branch of mathematical study: and thus it is, that all successful mathematical instruction converts the science into an instrument in the hands of a student, with which he investigates and deduces truths *for himself*. Science, in forming him, makes him not a passive recipient of knowledge, but an active discoverer of it; *and a successful applier of what is discovered to farther discovery, as well as to daily practice.*

5. That the student may commence, and continue with a fair prospect of success, let him at once be taught, that the vagrancies of the fancy, or even of the intellect, must be brought to one, as an incipient point from whence he may proceed with energy, according to the original bent of his thoughts, and so discipline himself as to compel his mind to devote itself solely and exclusively to the selected topic of study, until at his own bidding, he yields to the periodical and fixed season of remission.

Then, to start with clear views, let him be taught that the object of contemplation to the mathematician, is not whatever is susceptible of greater and less, but what is *measurable* (either in fact, or in conformity with a correct definable notion); and

that mathematics therefore, is not the science of magnitude in its most abstracted and general acceptation, but of magnitude that can be *measured*, that is, *referred to a unit*. It may indeed be regarded, and so should be explained, in taking the pupil over the general principles, (p. 1, 2, &c. vol. i.) as the *science* of mensuration, and which defines, regulates, and employs measures and units of measure[1].

This should be illustrated; First. By showing how numbers, money, weights, space, or extension, lineal, superficial, &c., time, motion, velocity, angular variation of direction, force, &c., become obviously susceptible of mathematical investigation.

Secondly. By showing that many things which are capable of the distinctions of greater and less, are not *yet* (and some perhaps may never be) brought within the range of *mathematical* determination; as pain, fever, intensity of colour of the same kind, of sound, &c.

Thirdly. By pointing the student's attention to

[1] It is not unworthy of attention, that the idea of motion is involved in the very conception of actual measurements of any kind: and hence that motion itself, so far from being extraneous to the metaphysical researches of pure mathematics, is more probably an essential element in them, however much it may be kept out of sight. This, however, is a consideration for preceptors in their more recondite inquiries, and not one with which a student of the first elements need perplex himself.

the interesting cor-relation between the extension of mathematical knowledge and improvements in the correctness of admeasurements, as beautifully exemplified in the Trigonometrical Survey of England and Ireland; the scientific contrivances for measuring a *base of verification*, improving the national weights and measures, &c.

6. In scientific teaching, the preceptor should well consider the character and functions of the intellectual faculty, *memory*. It has two conditions which must be carefully distinguished. Of these, the first is *remembrance*, viz., the power of *retaining* the impressions made: the second is *recollection*, viz., the power of *recalling* them. These, though altogether distinct, are often confounded. Many of the events of childhood, for example, may not occur to the recollection for half a century, and may hence be regarded as forgotten, and entirely lost to the mind; yet, frequent experience contradicts this notion: for it is no unusual thing, for the occurrence of a trivial event, at the very decline of life, to bring to mind, by the irrevocable law of *association*, some equally trivial occurrence of early youth; accompanied, at the same time, by the conviction that it never, from the very moment of its taking place, was before brought to recollection. Judicious mental discipline would, doubtless, diminish the frequency

of such intellectual phenomena, but they often occur.

7. Now, the grand characteristic of memory, rightly cultivated, is its UNIVERSALITY in virtue of which *nothing once remembered can ever be forgotten*, though it may lapse by a temporary slip, of greater or less duration; or though, in other words, it may not be *recollected* at pleasure, except the mind be under thorough discipline: and the concatenations of mathematical science furnish the best discipline, by bringing into full operation the principle of *association*, as well as the faculty of *attention*, and the habit of *logical arrangement*; all especially useful to military men, and all, if I do not mistake, more fully exemplified by well-educated military men, than by individuals in any other class of society[1].

8. That memory may accomplish all its valuable purposes, especially in the second condition, or that of recollection, it must rest upon *thorough comprehension* of the several subjects, without which it would be quite possible for a youth to commit the

[1] Dr. Hutton, in his best days as a preceptor, used to enforce his notion of the value of depositing scientific truths in the mind, conformably with the law of association, by a familiar illustration, "Sir, you must arrange the whole by scientific rules, and then so lodge it in the memory, that you can draw it out as you want it, just as a lady can draw out the thread or silk from her housewife, *either by a single thread, or by a skein*, as she pleases."

whole of Euclid accurately to memory, and yet not obtain a single correct geometrical conception, even though he might recal the entire series in its order and connection. The faculty should, also, be exercised, not so much in the remembrance of detached propositions, as in the power of recalling propositions arranged in groups or classes, so as to recur with facility in obedience to the law of association,—a principle of superlative advantage to the teacher who has well studied its nature and use. Such a teacher will always ascertain, even from the countenance of his pupil, when engaged in a demonstration, whether he is working from memory or from intelligence; and he must never let him proceed in the first case till by the interposition of his own instructions and illustrations he has secured that clear comprehension of the subject in hand, which will alone justify a farther advance. Such a teacher will, moreover, in the progress of his instructions, avail himself of one or other of the beautiful analogies and associations and generalizations which mathematics, more than any other science, (except, perhaps, chemistry) incessantly presents. He will also instruct his pupil how to avail himself of the advantages which the localities and intersections of a diagram, the prevailing symmetries in an equation, the laws regu-

lating the co-efficients, the exponents, &c., present; as well as point out the indications and tests by which such relations are detected and brought to use.

9. Intimately blended with this principle of association, or, indeed, immediately flowing from it, are the advantages of teaching every thing with a *prospective reference; uniformly* in the teacher's mind—*often*, as the occasion may serve, or as the character of the pupil may require, adequately unfolded to him. Thus, in explaining and enforcing the principles and practice of notation, fractions, composite factors, surds, &c., let the prospective reference to equations be sufficiently marked. So in exponential equations, to logarithmic and other series; so in geometry, to trigonometry, mensuration, mechanics, &c.; as, for example, Euc. vi. 33, with its obvious prospective reference to the measure of an angle in trigonometrical researches. And so, again, various inquiries in geometry, series, inequations, or inequalities, &c., manifestly suggest the consideration of *a limit*, and thus prepare the way for its thorough comprehension in the theory of the Differential Calculus. And, be it carefully observed, that many of these anticipatory suggestions may not only be made consistently with a strict adherence to scientific method, but so as to

stimulate curiosity and a spirit of inquiry; and thus at once to produce the love of scientific truth, and the best means of acquiring it.

10. Concurring in utility with the above, is the habit of ascertaining, by occasional *reviewing*, the extent and the firmness of the path already passed over. By adopting this plan, and following it judiciously, *time is not lost, but saved*, because of its tendency, in conjunction with the process of prospective reference, to awaken and keep alive that mental activity which effectually surmounts obstacles and ensures continuous progress. A class, or an individual, will, at any period of the course of study, be enabled, with the slightest effort, to recal whatever is duly and rightly acquired. Geometry, especially Euclid's Elements, presents peculiar incitements to the formation of this habit; and so, in its measure, does Algebra, in its clearly marked subdivisions. Indeed, the advantages of these revising processes are by no means confined to any one book, or any one scientific subject. By presenting the whole field of thought and inquiry at one view before the mind, it will strengthen the power of pursuing an extended range of argument or research; of examining and deciding upon a connected chain of reasoning; and will, in no mean degree, accustom and prepare the student to carry forward in his own mind a train of original

investigation: and this, let it never be forgotten, is the legitimate ultimatum of all scientific training.[1]

[1] Especial care and judgment are requisite in these reviewings and revisions, not to make them detrimental instead of beneficial. To ensure this, they must not be too frequent nor too long; and they must be directed far more to ascertaining that principles, and the principles of methods be rightly fixed, or to give them birth if they have not been correctly fixed there before, than to fatigue and disgust pupils by a tiresome repetition of rules. Teach youth to distinguish early and thoroughly between the memory of things and the memory of words; and in looking for formulæ, for example, to fix those most firmly in the mind that will each furnish a fund by useful and simple transformations.

M. *Lacroix*, in his valuable "*Essais sur l'Enseignement,*" makes some excellent observations immediately to this point, which I will quote. "Ce qu'il faut bien posséder, c'est la marche des méthodes, la valeur des termes techniques, l'intelligence des *idiotismes* de la langue, ou la faculté de saisir le sens des phrases et des formes d'expressions particulières aux principaux écrivains qui ont traité de la science, afin de pouvoir à la simple lecture comprendre leurs ouvrages, au moins ceux qu'on a étudiés, ou dont on pourra avoir besoin dans la suite; enfin, connaître la nature et l'enchainement des objets qu'ils contiennent, afin de pouvoir les consulter avec fruit lorsqu'il sera nécessaire.

"La mémoire la plus exercée n'atteint pas toujours ce but; la petitesse du cercle dans lequel sont nécessairement renfermés des objets appris par cœur, ne permet pas de mettre dans ces objets assez de variété, pour qu'il puissent offrir des examples des principales difficultés qu'on rencontre dans la lecture des livres.

"Puisque ce n'est pas un effort de mémoire que constitue le vrai savoir en mathématiques, et qu'il restreint plutôt les facultés qu'il ne les augmente; c'est donc à tort qu'on emploie un *examen oral et*

11. The practical application of the preceding remarks will be greatly facilitated by the recent

*par cœur*, pour s'assurer de la capacité des jeunes gens qui se livrent à l'étude de ces sciences. Aussi il est arrivé souvent que les hommes les plus instruits sont convenus de bonne foi qu'ils ne se croyaient pas assurés d'être reçus à un examen de ce genre, quoiqu'il portât sur des objets fort au dessous de leur connaissances. On a entendu, dans une des leçons qu'il a données à l'Ecole Polytechnique, Lagrange lui-même le dire avec cette modestie qui le caractérise si éminemment. En effet, contens de posséder l'esprit des méthodes, et de savoir revenir sur les détails lorsqu'ils leur deviennent nécessaires, les géomètres n'entreprennent pas de les confier à leur mémoire ; ils se gardent bien de se condamner à un travail fastidieux, qui émousserait en eux l'esprit d'invention et de recherche. Les professeurs eux-mêmes, qui parcourent successivement ces détails, ne cherchent à se rappeler que ceux dont ils ont besoin dans un intervalle de tems très-limité. Comment donc peut-on demander avec justice aux disciples ce qu'on n'exigerait pas du maître ? Ignore-t-on le tems qu'on leur fait perdre à repasser, osons-le dire, à rabâcher sans cesse la matière d'un examen, pour se tenir en haleine et se préparer à répondre en même tems sur tout ce qu'ils ont appris ? Croit-on que le dégoût qui suit nécessairement un travail aussi monotone, n'arrête pas le plus souvent les progrès des jeunes gens au terme où finit leur examen, ne les porte pas quelquefois à se débarrasser promptment la tête de connaissances qu'ils n'ont péniblement acquises que pour en faire parade un seul jour, parce qu'ils n'ont pas senti ce charme que la variété jette sur des études qui présentent des objets nouveaux qu'on n'épuise pas ? Aussi beaucoup d'entr'eux, guidés quelquefois en ce point par leurs maîtres, étudient le goût, les habitudes des examinateurs, cherchent exclusivement ce qui peut abréger et adoucir l'épreuve

extension of the pre-requisites for admission into the Royal Military Academy. It is now no longer necessary to commence with the rudiments of practical arithmetic, and then to proceed to algebra and geometry. A cadet, or other student, who enters an institution prepared adequately, or even moderately, up to the commencement of quadratic equations, and through from one to four books of Euclid's Elements, may, manifestly, be otherwise dealt with. After the master who first receives him has ascertained the precise extent and stability of his acquirements, including, of course, the *com-*

qu'ils doivent subir, et rejettent comme inutile pour eux tout ce qui ne s'y rapporte pas."

After the above, what follows, which occurs two or three pages onwards, will be read without surprise.

"Après avoir par ce moyen brillé le jour de leur examen, le plus grand nombre des aspirans, devenus gardes de la marine, perdaient au bout de quelques campagnes de mer la théorie qu'on leur avait inculquée, pour ainsi dire, sans leur participation, et n'étaient point capables de l'apprendre par eux-mêmes dans les livres. C'est ainsi que j'ai vu un garde de la marine, *reçu le premier des trois ports, ne pas se rappeler la théorie arithmétique des fractions, trois ans après être sorti de l'école.*"

The aspirants of artillery at the school at Metz were prepared by a like process, and with a like failure of success, until the formation of the Polytechnic School, and the united skill and genius of Monge, Hachette, Carnot, Lacroix, Legendre, Prony, Poisson, &c., gave new life to science, and showed the mistaken folly that had too long preserved this order of things in seminaries of education.

*prehension* as well as the *recollection*, of the definitions[1] in those departments of mathematics which he has already gone over, he will so teach him the *principles*, together with the *practices* of arithmetic and algebra, as to cause them mutually to confirm and illustrate each other; while, by making a due appropriation of the times respectively devoted to geometry and to algebra, &c., the two main departments may be made so to move on *pari passu*, that each, while it furnishes its peculiar stores to its own specific purposes, may contribute its full share of illustration, or of proof, to the other; and thus the foundations for the superstructure of science may be spaciously as well as safely laid.

12. The details of such an arrangement as is above adverted to, must grow with the lapse of time, and from the working of the whole system of the Institution, (in which much, in addition to mathematical instruction, rightly constitute essential portions,) and cannot, therefore, be prescribed here. All that it is proposed farther to attempt in this paper is, by assuming the first volume of the eleventh or any later edition of Hutton's

[1] See farther, art. 25. A few of the observations here occurring for a page or two, though originally applicable to the Royal Military Academy, in 1837, may, obviously, with slight modifications to suit existing cases, be rendered useful in various other seminaries, public and private.

Course of Mathematics, and R. Simson's Euclid, as the text-books employed in instruction, to mensuration inclusive[1], to direct the attention of the mathematical masters to *specimens* of the several points to be ascertained, enforced, or explained, in giving instructions upon those works respectively. It must be left to their knowledge, judgment, and experience, to carry out the whole plan, agreeably to the views and principles elucidated herein, or with such modifications only, as may manifestly subserve the same general purposes, and as manifestly lead to the same ulterior results.

13. Let each cadet (or other student) be early taught the essential distinction between an inert and passive reception of the truths of science, (as though the mind were a mere vessel,) and the active search after it, with a view to the full possession of all its advantages, in extending and sharpening the intellectual powers, as well as in directing and improving the practical applications. To this end, let him be taught that the text-books

[1] It is proposed to connect *Analytical Trigonometry* only very slightly with the Trigonometry of the first volume, that it may be treated more extensively, both as to its researches and its uses, in a subsequent part of the course. The same procedure may often be advantageously adopted in seminaries, where clear, sound instruction for practical purposes, is more the object than to produce eminent mathematicians.

employed, whether in the earlier or the more advanced parts of the course, though regarded on the whole as best adapted to the purposes for which they are respectively employed, and in the main correct, are never to be consulted as *infallible.* The conviction which they carry is to be sought in *the evidence* which they produce. The student is to be taught how to seek out, to weigh and estimate that evidence, to ascertain what portions of it may be unhesitatingly received, and made his own; what portions rejected, or what may be taken with a kind of suspended acquiesence, until farther light renders it clear, whether they may be safely adopted, or must be set aside altogether. He will thus become capable of distinguishing between theorizing without intellect, and by means of it, and, in consequence, between performing an operation mechanically as a mere machine, or working by a rule which springs from intelligence.

These distinctions, rightly made, are *the infallible key to ultimate success;* with a view to which, the instructor should, as he proceeds, connect with the instructions supplied by the text-book, all such observations, corrective or elucidatory, and enforce all such deductions, improvements in method, and extensions of principle, as will either illustrate and confirm the subject in hand, or receive advantageous applications in subsequent portions

of the course. Let him, however, all along, while he impresses upon the student the extreme importance of continuous and close attention, as the main pre-requisite of success; on his own part never lose sight of the essential distinction between what may simply try the *patience*, and what will effectually try the *head*, and exercise and develope the *faculties* of his pupil; and thus abstain perpetually, with the utmost circumspection, from all which would merely perplex, without yielding commensurate practical advantage.

The remarks which follow, are throughout meant to be more or less subservient to the purposes here hinted at. To prevent mistakes, I repeat, that they are presented as *specimens*, and not as embracing the whole of the particulars, which cannot be neglected without serious detriment to the student.

14. He should never be permitted to slip over a note, a corollary, or a scholium, in any part of *Hutton* or of *Euclid:* and the same is manifestly applicable to the beneficial use of any other text-book. The master will, of course, not detain him upon a note or observation of little or no value. He will teach him to discriminate; to despatch some, after listening to a few oral observations, to attend more thoroughly to others; and, where they contain or require a demonstration, either to

enforce or supply it, or explain why it had better be referred to some more appropriate place.

15. The student should be accustomed from the very first, to employ the symbols $+$, $-$, $\times$, $\div$, $:$, $::$, $\therefore$, $\surd$, $\sqrt[3]{}$, &c., without reserve. They often suggest simplifications in the operations, which are of great practical value, especially in reductions: they also serve to render easily intelligible some of the demonstrations to rules in Arithmetic; and they facilitate the habit of generalization, and of making a ready transition from one to another, of the kindred processes in Arithmetic and Algebra.

In Arithmetic and Algebra, for example, the processes for combining and transforming fractions are the most troublesome, but are susceptible of the most valuable modifications, by attending to the use of symbols; often, indeed, by the skilful interposition of the dot (.) in multiplication.

16. While the attention of the student is called to the signification and value of *symbols,* let no opportunity be omitted of explaining the nature and character of the *things* connected, or denoted, by those symbols; and let both the *distinction,* and the *connexion* between *number* and *magnitude,* one of the most difficult points in the investigations of mathematics, be elucidated and enforced on every available opportunity.

17. Geometry can, in truth, proceed but a little way without Arithmetic, and the connexion was first made by Euclid, in his fifth book, a confessedly obscure book to a learner; and yet, in the pure Greek editions, contemplated logically, one of the most unobjectionable treatises that ever was written [1]. The subject is continued by Euclid in four subsequent books, unfortunately inserted but in very few of the modern editions. He who permits a student to enter upon Euclid's fifth book, without showing that its topics and its reasoning are altogether different from what has preceded, *may enforce the practice of getting by rote, but does not teach geometry.*

When a student is tolerably versed in the ordinary operations and principles of Arithmetic and Algebra, (as the Woolwich cadets must now be on admission,) he is placed on vantage ground, of which the master must, at once, show him how to avail himself; and he cannot do it better, than by making him thoroughly comprehend, that the doctrine of PROPORTION is, in truth, the basis of mathematics, and by exemplifying and enforcing

[1] A very neat, compendious, and accurate Greek edition of Euclid, was published *by E. F. August,* at Berlin, in 8vo. Vol. I., published in 1826, contains the first nine books. Vol. II., in 1829, contains books ten to thirteen, with variations and other notes, and a life of Euclid. I should rejoice if this edition were better known than it yet seems to be in England.

this, by the joint aid of Arithmetical and Algebraic considerations, still farther illustrated and confirmed by the doctrine of Euclid's fifth and sixth books.

18. The student should first be taught to distinguish number, that is, *abstract number*, and *quantity or magnitude;* the first regarded as including the notion of times or *repetitions*, independently of the things counted or repeated; the second considered as whatever is made up of *parts*, not differing from the whole in any respect, but that of *being less:* and then, if he be put upon the inferences, susceptible of being deduced from the consideration of a part and the whole, he will perceive that they are only two: viz.

(1.) The part is less than the whole.

(2.) The whole is greater than a part [1].

He will thus learn to distinguish between a *definition* and an *axiom;* and may immediately be

[1] "There is no other quantity from which number may not be *abstracted:* therefore, first, number is *perfectly abstract;* secondly, *it is the only thing which is so.*

"Again, we conclude, that as every thing specific has been excluded in obtaining it, it is a question whether or not it is rightly denominated *a species of quantity.*

"Lastly, notwithstanding its essential difference, it is intimately and equally connected with every species of [measurable] quantity, as being the only natural *measure* and representation of its magnitude."

taught the joint use of definitions and axioms, and their consequent tendency to give cogency and certainty to that *progressive* reasoning which distinguishes the mathematical from the ordinary logic.

In the appropriate language of Professor Whewell, "the chains of the logician generally consist only of two or three links. In mathematics, on the contrary, every theorem is an example of an extended progressive chain; every proof consists of a series of assertions, of which each depends on the preceding, but of which the last inferences are no less evident, or no less easily applied, than the simplest first principles. The language contains a constant succession of short and rapid references to what has been proved already; and it is justly assumed, that each of these brief movements helps the reasoner forward in a course of infallible certainty and security. Each of these hasty glances must possess the clearness of intuitive evidence, and the certainty of mature reflection; and yet must leave the reasoner's mind entirely free, to turn instantly to the next step of his progress."

The faculty of pursuing such processes, readily and safely, is of inestimable value; and, if I do not greatly mistake, an *intelligent* student may, by judicious scientific instruction, be *now* placed in full possession of this faculty, within a month or

six weeks of his joining any institution, the formerly mentioned pre-requisites of knowledge being possessed.

(*a.*) To this end, let him at once be taught, that in all cases where the relation of greater and less cannot be ascertained, either there is *equality* as to sense, or the inquiry does not come within the province of mathematics. That in many other cases, even in ordinary Arithmetic, which are, simply for practical purposes, referred to specific rules, a scientific classification refers them to the province of ratios and proportions. It is thus, with regard to *reductions*, to *fractions*, to *multiplication*, and *division*, &c. For, when we reduce money, lineal measures, superficial measures, measures of capacity, from one denomination to another, we simply apply the measuring unit, either by repetitions or deductions, according to fixed and known ratios: when we say that one number or quantity is $\frac{1}{12}$, or $\frac{1}{20}$ of another, we mean nothing else, than that one is to the other as one to twelve, or as one to twenty. When we multiply or divide a number or a quantity by fifty, for example, we simply ascertain another number or quantity, which is to the former as fifty to one, or as one to fifty. And so of other cases.

19. This is not a refined technical distinction, but one of the utmost practical value; since it transfers the whole of Arithmetic, and all that de-

pends upon it, within the scope of the doctrine of proportions; and thus facilitates the theoretical comprehension of the reasons of the various processes, while it confers upon the pupil the capacity of generalizing with safety and success [1].

[1] To impress this fully upon the mind, a skilful preceptor will find it important and useful to select, for the investigation of his pupils, a few of the questions often given at the very commencement of simple equations, and show how they not only serve to elucidate the properties of proportions, but admit of clear solution from simple arithmetical principles.

For example:—

A father is 40 years old, and his son 12; in what time will the age of the father be triple that of the son?

When one number is triple another, the difference of these numbers is manifestly double the less; but whatever be the ages of father and son, the difference of their ages will always remain the same. In the present instance the difference is 28 years, and 28 is double of 14; therefore the son must be 14 years of age, or 2 years must elapse before the father's age be triple that of the son.

Were it required to find in what time the father's age would be double that of the son; then 28 years, the difference of their ages, must be the age of the son; and therefore, it will require 16 years to elapse, before their ages be in the specified proportion.

Again, if we wanted to know when the father's age was 8 times that of the son; then 28, the difference of their ages, must have been 7 times the son's age; and therefore the son was just 4 years old when their ages were in the above-mentioned proportion; and 8 years have now elapsed, since the father's age was 8 times that of the son.

Here, as often, there will be an advantage in appealing to reason, rather than in calling up a rule.

20. With this understanding, let the student be taken over the subjects of proportions, fractions, and progressions, both in the Arithmetic and the Algebra, in scientific connexion, with occasional hints and aids, from p. 317—322, vol. i., edit. ii., (On Ratios and Proportions,) or supplied by the master, as the occasion may require, either from his own resources, or from the well-known portions of *Wood*, *Bridge*, *Garrett*, and *Hind*, devoted to kindred inquiries. When neither of these is at hand, the following hints may be found useful.

Let the student be taught:

(*a.*) That the ratio of two quantities of the same kind, is equal to the ratio of the abstract numbers which represent the number of times that any common measure can be repeated in each, or applied to each.

(*b.*) That a careful distinction must be made between equal and unequal ratios, as well as their representatives.

(*c.*) That four quantities are said to be proportionals, when the first is the same multiple, part, or parts of the second that the third is of the fourth.

*Note.* This may be regarded as an Algebraical definition of proportion, of which the most useful consequences are traced in the "Course," and may, at the master's pleasure, be otherwise deduced.

It should be farther shown,

(*d.*) That if magnitudes be proportionals, according to the Algebraical definition of proportion, they are also proportionals, according to the geometrical, or Euclidean definition.

And the converse, and hence—

(*e.*) That whatever consequences are strictly deducible from one of these, may be legitimately applied to the other.

(*f.*) That, if $a : b$ and $x : y$, be equal ratios, the fractional representative of the former shall be *equal* to that of the latter: that is $\frac{a}{b} = \frac{x}{y}$.

[The following *indication* of a proof may be readily filled up orally, with references to the principles implied: and so it will be with regard to some subsequent indications of proofs.]

$$a : b = x : y$$

$$\frac{a}{b} : 1 = \frac{x}{y} : 1$$

$$\therefore \quad \frac{a}{b} = \frac{x}{y},$$

since each has the same ratio to unity.

(*g.*) Let the above be blended with the observations and notes at pp. 46, 79, 83, &c., vol. i., Hutton, and let the advantages and disadvantages of representing ratios by fractions, and proportions by equal fractions, be carefully pointed out.

(*h.*) When four numbers are direct proportionals, the product of the extremes is equal to that of the means.

Thus:—

| | |
|---|---|
| If | $a : b :: c : d$ |
| then, | $a : b = c : d$ |
| also, | $d : c = d : c$ |
| $\therefore$ | $ad : bc = cd : cd$ |
| But | $cd = cd, \therefore ad = bc$ |

*Cor.* Hence $a = \frac{bc}{d}$, $d = \frac{bc}{a}$, $b = \frac{ad}{c}$ and

$c = \frac{ad}{b}$ $\Big\{$ whence the rules of proportion in arithmetic.

[Let it be remarked that the ordinary rules for stating and working the *Rule of Three Inverse,* in many books of arithmetic, are in violation of this principle.]

(*i.*) When three quantities are proportional, *i.e.*, when the first is to the second as the second to the third, the ratio of the first to the third is equal to the duplicate ratio of the first to the second.

(*k.*) The duplicate ratios of equal ratios are equal to each other.

(*l.*) That ratio which is compounded of three equal ratios is either the triplicate ratio, or is equal

to the triplicate ratio of one of those of which it is compounded[1].

(*m.*) When four terms are in continued proportion, the ratio of the first term to the fourth is equal to the triplicate ratio of the first term to the second.

Let the four terms be $a$, $b$, $c$ and $d$: that is,

$$\begin{array}{llcl}
\text{Let} & a : b : c : d & & \\
\text{Then} & a : b & = & a : b \\
 & b : c & = & a : b \\
 & c : d & = & a : b \\
\therefore & abc : bcd & = & a^3 : b^3 \\
\text{But} & abc : bcd & = & a : d \\
\text{Because} & abc & = & a(bc) \\
\text{And} & bcd & = & (bc)d \\
\therefore & a : d & :: & a^3 : b^3
\end{array}$$

(*n.*) To multiply by the component parts of a composite number, is the same in effect as to multiply by the composite number itself, and *vice versâ*.

EXAMPLE.

To multiply 23 by 15.

[1] Several propositions of a kindred nature to these are neatly and perspicuously demonstrated by Mr. Garrett, in his Essay on the Doctrine of Proportion.

Here—

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| | 1 | : | $\underline{15}$ | :: | $\underline{23}$ | : | product. |
| | 1 | : | $\underline{23}$ | :: | $\underline{15}$ | : | product. |
| × 5 .... | 5 | : | 115 | :: | 15 | : | product. |
| × 3 ... | 15 | : | 345 | :: | 15 | : | product. |

But $15 = 15 \therefore 345 =$ product.

$\therefore 23 \times 5 = 115$, and $115 \times 3 = 345 =$ product.

Here, unity is to the multiplier as the multiplicand to the product; and, by alternation, unity is to the multiplicand as the multiplier is to the product; consequently, 5 times unity is to 5 times the multiplicand, as the multiplier is to the product; and 3 times 5 times, that is, 15 times, unity, is to 3 times 5 times, or 15 times the multiplicand, as the multiplier is to the product.

A similar proof may be applied to any other case: an analogous one to division; and like considerations furnish the best reason for the rule of the signs in multiplication of algebra.

(*o.*) The following synopsis of proportion may assist the recollection, after studying the 5th Book of Euclid.

SYNOPSIS OF PROPORTION.

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| If | A | : | B | :: | C | : | D |
| Permutando | A | : | C | :: | B | : | D |
| Invertendo | B | : | A | :: | D | : | C |

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| Componendo | $A+B$ | : | $B$ | :: | $C+D$ | : | $D$ |
| Dividendo | $A-B$ | : | $B$ | :: | $C-D$ | : | $D$ |
| Convertendo | $A$ | : | $A+B$ | :: | $C$ | : | $C+D$ |
| or | $A$ | : | $A-B$ | :: | $C$ | : | $C-D$ |
| Also, if | $A$ | : | $B$ | :: | $D$ | : | $E$ |
| and | $B$ | : | $C$ | :: | $E$ | : | $F$ |
| then, | | | | | | | |
| Ex æquo ordinate | $A$ | : | $C$ | :: | $D$ | : | $F$ |
| Also, if | $A$ | : | $B$ | :: | $E$ | : | $F$ |
| and | $B$ | : | $C$ | :: | $D$ | : | $E$ |
| then, | | | | | | | |
| Ex æquo perturbate | $A$ | : | $C$ | :: | $D$ | : | $F$ |

(*p.*) Let the student receive a clear explication of the reason why in Euc. v. def. 3, there is a definition of ratio which is apparently simple; and then, in def. 5, an apparently complex *test* of equality of ratios.

(*q.*) Let his mind be also fixed upon the distinction between the ordinary acceptation of the word *part*, and that adopted by Euclid in the 5th book, and to the consequences of that distinction.

[In common language, a *part* of a magnitude means any portion of it whatever. Euclid limits the meaning to that portion of the magnitude which is contained in it a number of times exactly. Thus, in common language, 7 is *a part* of the number 24;

while, according to Euclid, 2, 3, 6, 8, 12, (NOT 7,) are parts of 24.

Why did Euclid take this view, and what are its consequences, must be clearly explained. One of these consequences is the comparatively modern distinction between *arithmetical* and *geometrical* Proportion and Progression; or between *equidifferent* and *equirational* progression, as they are now often called.

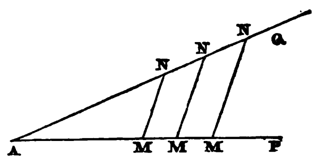
Also,

(*r*.) Why, in the demonstrations of Euclid v., *right lines* are assumed as the representatives of all such magnitudes as are brought into consideration in that book.

Let the student be taught to distinguish between the *terms* of a ratio and the ratio itself, by being shewn—

(*s*.) That *the terms* of a ratio *may vary* through all degrees of magnitude, and yet *the ratio itself continue invariable*.

Let AP, AQ, be two indefinite right lines, intersecting in a given angle A, and on any point M in AP, draw




MN to make any given angle with AP, and till it meets AQ in N. Let the point M be supposed to move along AP, so that AM and MN (always

making the same angle with AP) may increase and decrease. The lines AM and MN in this case may vary through all degrees of magnitude, may become greater or less than any assigned line; yet the ratio of AM to MN remains invariably the same, because the triangles AMN, AMN, are always similar. (Euc. vi. 2.)

On the other hand, it should also be proved—

(*t.*) That, in numberless instances, although changing *the terms* does change *the ratio,* yet the variations *cannot* go beyond certain limits, even though the terms themselves should increase or decrease, *in infinitum.*

[Of this the following example, which is very instructive, is quoted from *Saunderson's Algebra,* now becoming a scarce book.]

Let $x$ be any varying quantity: make $4x^2 + 3x = A$, and $2x^2 + x = B$; then will A and B evidently be varying quantities, as depending upon $x$. When $x$ vanishes, or becomes nothing, A and B will both vanish; and when $x$ is infinite, both A and B will be infinite. Now, I say that the ratio of A to B, while $x$ decreases *in infinitum,* approximates to the ratio of 3 TO 1, as its limit. For, 1st, A is to B as $4x^2 + 3x$ to $2x^2 + x$, or, as $4x + 3$ to $2x + 1$. Here it is obvious that as $x$ decreases, the quantities $4x$ and $2x$ decrease, and consequently the ratio of $4x + 3$ to $2x + 1$, or of A to B, ap-

proaches to that of 3 to 1. 2ndly. The ratio of A to B can never reach or exceed that of 3 to 1. For $6x^2 + 3x$ is to $2x^2 + x$ as 3 to 1; but $4x^2 + 3x$ is a less quantity than $6x^2 + 3x$; therefore $4x^2 + 3x$ is to $2x^2 + x$, or A is to B, always in a *less* ratio than that of 3 to 1. Lastly. The ratio of A to B will approach nearer to that of 3 to 1, than by any assigned difference: For, in the terms of this ratio $4x + 3$ and $2x + 1$, the varying parts $4x$ and $2x$ may, by diminishing $x$, become less than any assigned quantity, while the other parts, 3 and 1, remain the same: therefore the ratio of A to B will, in this case, approach nearer to the ratio of 3 to 1 than by any assigned difference.

Again, in like manner, the ratio of A to B, while $x$ increases *in infinitum*, approximates to the ratio of 2 TO 1 as its limit. For, 1st. Since A is to B as $4x + 3$ to $2x + 1$, or, as $4 + \frac{3}{x}$ is to $2 + \frac{1}{x}$; it is easy to see that, as $x$ increases, the quantities $\frac{3}{x}$ and $\frac{1}{x}$ decrease, and consequently the ratio of $4 + \frac{3}{x}$ to $2 + \frac{1}{x}$, or of A to B, approaches to the ratio of 4 to 2, or of 2 to 1. 2ndly. The ratio of A to B can never be less (that is, never be nearer to the ratio of equality) than the ratio of 2 to 1. For $4x^2 + 2x$ is to $2x^2 + x$, as 2 is to 1; but $4x^2 +$

$3x$ is a greater quantity than $4x^2 + 2x$; therefore, $4x^2 + 3x$ is to $2x^2 + x$, or A is to B, always in a greater ratio than that of 2 to 1. Lastly, the ratio of A to B will approach nearer to that of 2 to 1, than by any assigned difference. For in the terms of this ratio $4 + \frac{3}{x}$ and $2 + \frac{1}{x}$, the variable parts $\frac{3}{x}$ and $\frac{1}{x}$, may, by increasing $x$, become less than any assigned fraction, while the parts 4 and 2 remain the same: therefore the ratio of A to B will, in this case of the indefinite augmentation of $x$, approach nearer to the ratio of 4 to 2, or that of 2 to 1, than by any assigned difference.

Here then it is manifest that, though diminishing $x$, and consequently diminishing the terms A and B, *increases* their ratio, and, contrariwise, increasing those terms by increasing the quantity $x$, *decreases* their ratio; yet, there is *a limit* both to the increase and decrease of the ratio itself, although there is *none* to the terms which compose it.

[Such an example is valuable, if properly explained and enforced; because it serves to shew that the doctrine of limits, instead of being very recondite, and solely intelligible at an advanced stage of the mathematical course, flows naturally and clearly from the earliest elements.

Another example of a similar nature, such as that furnished by the comparison of $x + d$, and $x$, where $x$ alone is variable, may suffice to make this matter thoroughly comprehensible.]

(*v.*) An unreflecting pupil is sometimes found to slide into the belief that adding the same number, $n$, to the terms of a ratio or of a fraction, $\frac{a}{b}$, causes no change in its value.

Let him be taught to convince himself of the contrary, by tracing out a simple process.

Thus let him prove 1st, that if $a < b$, then $\frac{a+n}{b+n}$ is $> \frac{a}{b}$; and 2ndly, that if $a > b$ then $\frac{a+n}{b+n}$ is $< \frac{a}{b}$.

In the first case, $\frac{a}{b} = \frac{ab + an}{b^2 + bn}$, while $\frac{a+n}{b+n} = \frac{ab + bn}{b^2 + bn}$. Here, the denominators being equal, the fraction with the greatest numerator is greatest, that is, since $ab + bn$ is evidently $> ab + an$; therefore $\frac{a+n}{b+n}$ is $> \frac{a}{b}$.

The second case may be proved similarly.

In addition to the preceding, let the instructor clearly mark the distinction between *abstract* and

*concrete* number; as well as those between *repetition*, varieties of *length, surface, capacity,* and *opening*, and show how either of these may suggest the idea of *ratio*, while a periodical function, as that of an arc extending through from 1 to 5, 7, 9, &c., quadrants, suggests a still more remote class, and requires a more refined analysis; he will then have opened a wide and fruitful field, and must point out to the student over what portions he may make his excursions freely and safely, as well as direct him to a barrier where he must pause, and make these deductions, viz.:—

(*w*.) 1. Abstract numbers are certain ratios.

2. Abstract fractions are certain other ratios.

3. All possible ratios are not found among numbers and fractions.

Whence it follows—

4. That primary arithmetic, or that which relates to abstract and definite numbers, though it may be, as far as it goes, a theory of ratios and proportions, is not a theory of *all* ratios and proportions, nor are its operations such as can be performed upon *all* ratios or proportions.

In other words—

5. We in many cases safely apply the conception of proportionality to examples in which one quantity is neither multiple, part, nor parts (*c*) of another; as when we infer that the diagonals of two

squares are proportional to their sides, or the diagonals of two rectangles, whose sides are 5 and 7, and $5n$ and $7n$, are proportional to the sides 5 and $5n$, or 7 and $7n$.

6. In such cases the deductions, though correct, go beyond the limits of the definition. How, then, is the ground to be made secure?

Thus will the student be brought to the inevitable consideration of *incommensurables*, a topic with which the master should make him thoroughly conversant before he quits Euclid's 5th book.

21. Another subject in which great advantages will flow from a systematic junction of the arithmetical and algebraical considerations and processes, is that of *composite factors* and *common measures*, whether taken with regard to fractions simply, or in reference to their ultimate bearing upon the solution of equations.

Here the student, after duly attending to the definitions at p. 3, vol. i., should be referred to the rules and notes at pp. 41, 43, 44, (case 1, 3. Fractions) committing to memory the criteria of divisibility by 2, 5, 6, 8, 11, &c., and pursuing the inductions which led to those criteria. Let him be set to find out, whether or not there is a condition in the use of criterion 7, not specified. [That condition being, in fact, this: if the sum of the digits in the *odd* places is *not* equal to the sum of the

digits in the *even* places, still, when their *difference* is divisible by 11, the entire number is divisible by 11.]

He should then be required to collect, and, where necessary, seek the reasons of, a few other definitions and propositions: such, *ex. gr.*, as—

(*a.*) One quantity is said to *measure* another, when it can be taken from the other any number of times, or when it can be applied to the other any number of times, without leaving a remainder. Thus, 5 measures 5; 3 measures 12; 7 measures 35; &c.

(*b.*) One number is said to be a *common multiple* of several *less* numbers, when it is measured by each of the less.

Thus 12 is a common multiple of 6, 4, 3, 2, and 1.

(*c.*) One number is said to be a *common measure* of several numbers, when it measures each of them. Thus, 3 is a common measure of 3, 6, 9, 12, 15, &c. An hour is a common measure of a day, and of a week; but not, strictly speaking, of a year.

(*d.*) A number is said to be *absolutely prime* when it admits of no measure but itself and unity. As 7, 11, 13, 17, 19, 29, &c.

(*e.*) Numbers are said to be *relatively prime*, or prime to each other, when they admit of no common measure but unity. Thus 5 and 7 are both absolutely and relatively prime; 14 and 15 — *relatively*, but neither of them *absolutely* prime:

for 2 and 7 measure 14, but neither of them will measure 15; 3 and 5 measure 15, but neither of them will measure 14.

(*f.*) A number is called *composite* when it admits of more measures than itself; and unity, and those factors which produce it by multiplication, are called its *component parts*, or *composite factors*.

(*g.*) When one number measures another, it will measure *all multiples* of that other.

(*h.*) When one number measures several, it will measure their *sum*.

(*i*). When one number is a common measure of two numbers, it will measure their *difference*.

(*k.*) These being duly comprehended, let a simple example be worked according to the rule at p. 40.

EXAMPLE.

Find the greatest common measure of 124 and 280.

```
1st step—   124) 280 (2
                 248
                 ———
2nd step—         32) 124 (3
                       96
                       ——
3rd step—              28) 32 (1
                           28
                           ——
4th step—                   4) 28 (7
                               28
                               ——
```

Here 4 is the common measure[1].

That the number 4, just found, must be a *common measure* of 124 and 280, may be thus shewn :—

1. Because 4 measures 4 and 28—that is, itself and the last dividend—4 must measure 32 (*h*) the *sum* of 4 and 28.

2. Because 4 measures 32, it must measure 96 (*g*), a *multiple* of 32; and since 4 measures 28 and 96, it must measure 124 (*h*), their sum.

3. Because 4 measures 124, it must measure 248 (*g*), a *multiple* of 124; and since 4 measures 32 and 248, it must measure 280 (*h*), their sum: consequently, 4 is a common measure of 124 and 280, the proposed numbers.

Farther—That the number 4, found as above, must be the *greatest* common measure of 124 and 280, may be thus shewn :—

[1] Expedients, which will readily suggest themselves after a little practice, will often abridge the operation. Thus, in the above example, carry on the second step, instead of subtracting 3 times 32 from 124, by taking 124 from 4 times 32, and making use of the difference, as below :—

```
124) 280 (2
     248
     ---
      32) 124 (4
          128
          ---
            4) 32 (8
               32
               --
```

The common measure is 4 as before.

If possible, let any assumed number, as 5, greater than 4, be a common measure of 124 and 280.

1. If 5 measure 124, it must measure 248 (*g*) a multiple of 124; and if 5 measure 248 and 280, it must (*i*) measure 32, their *difference.*

2. If 5 measure 32, it must measure 96 (*g*), a multiple of 32; and if 5 measure 96 and 124, it must (*i*) measure 28, *their* difference.

3. If 5 measure 28 and 32, then, according to (*i*) it should measure 4, *their* difference; that is, a greater number should measure a less; but that is impossible; consequently, the supposition *that* 5 *may be a common measure of* 124 *and* 280, from which this absurd inference would follow, cannot be true. And the same may be shewn of any other number greater than 4.

A similar process, applied to two or three other cases, will, without a more refined investigation, produce a conviction of the truth of the method that cannot be shaken. It will farther serve to produce the conviction, that reasonings soundly conducted upon the principles of numbers alone, may lead to *infallible results*; and, therefore, that the use of numbers is not restricted to mere computations. [Other illustrations occur in Garrett On Proportion.]

(*l*). Algebraically, also, the truth of the usual method of finding the greatest common measure,

may be established after a process here briefly hinted.

Suppose the operation to stand thus:—

$$
\begin{array}{l}
b)\ a\ (p \\
\underline{pb} \\
\quad c)\ b\ (q \\
\quad \underline{qc} \\
\qquad d)\ c\ (r \\
\qquad \underline{rd} \\
\qquad\ \ \underline{o}
\end{array}
$$

then $c = rd$; $b = qc + d = (qr + 1)\,d$; $a = pb + c = (p + pqr + r)\,d$.

$\therefore$ $a$ and $b$ are both divisible by $d$, viz., by the units in $(p + pqr + r)$, and $(qr + 1)$, respectively; that is, $d$ is a common measure of $a$ and $b$.

It is also their *greatest* common measure: for if not, let $x$ be a greater, and suppose—

$$\frac{a}{x} = m\,;\ \frac{b}{x} = n$$

$$\therefore c = a - pb = mx - pnx$$

$$d = b - qc = nx - qmx + pqnx$$

$$\therefore \frac{d}{x} = n - qm + pqn\,;$$

that is, $x$ measures $d$; a greater quantity measures a less, which is absurd. $\therefore$ $d$ is the *greatest* common measure of $a$ and $b$. Q.E.D.

[Let the attention of the student be drawn to the evident analogy between these two methods, the arithmetical and the algebraical; and if he have developed by this time a turn for investigation, let him be employed upon processes of his own in which the analogy shall be drawn still closer.]

(*m.*) The greatest common measure is not *necessarily* the greatest numerical common measure of the numbers resulting from assigning numerical values to the letters comprehended in the expressions. For example, the greatest common measure of $a^2 - c^2$ and $ab + bc$ is $a + c$. But, if $a = 7, b = 6$, and $c = 3$, the greatest common measure of $49 - 9$ and $42 + 18$, is not $7 + 3$, but $2(7 + 3)$.

(*n.*) The divisors in the rule, note pp. 43, 44, ought to be restricted to prime numbers, otherwise it may happen that some factor is unnecessarily repeated.

Thus, $5 \times 2 . 3 \times 2 . 5 . 7$, is not the least common multiple of 50, 30, and 42, it containing an unnecessary factor, 5.

The ambiguous result may be avoided by restricting the divisors to the primes, as above, or, by employing and exhausting the lowest common divisors before proceeding to the higher ones.

Thus, in 2, 10, 10, 10, 75, 150, 4, if we commence with 10, we have—

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 10 | 2, | 10, | 10, | 10, | 75, | 150, | 4 |
| 5 | 2, | 1, | 1, | 1, | 75, | 15, | 4 |
| 3 | 2, | 1, | 1, | 1, | 15, | 3, | 4 |
| 2 | 2, | 1, | 1, | 1, | 5, | 1, | 4 |
| | 1, | 1, | 1, | 1, | 5, | 1, | 2 |

$10 \times 5 \times 3 \times 2 \times 5 \times 2 = 3000$, common multiple.

But commencing with 2, we have—

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 2 | 2, | 10, | 10, | 10, | 75, | 150, | 4 |
| 5 | 1, | 5, | 5, | 5, | 75, | 75, | 2 |
| 5 . 3} 15 | 1, | 1, | 1, | 1, | 15, | 15, | 2 |
| | 1, | 1, | 1, | 1, | 1, | 1, | 2 |

$2 \times 5 \times 15 \times 2 = 300$, least common multiple.

(*o.*) Even this labour, simple as it is, may often be shortened by a table of prime factors, such as *Barlow's* or *Lidonne's*, (the latter being far the most copious,) or by practice: thus—

$$3366 = 2 \,.\, 3 \,.\, 3 \,.\, 11 \,.\, 17$$
$$4590 = 2 \,.\, 3 \,.\, 3 \,.\, 3 \,.\, 5 \,.\, 17$$

(*p.*) Let the mutual relation between the greatest common measure, and the least common multiple (G.C.M.) and (*l.c.m.*) of two numbers be shewn; and the use of that relation in reducing fractions, in simplifying and solving equations.

[Let $m$ be *any* common measure of $a$ and $b$, and suppose—

$$\frac{a}{m} = Q\,;\ \frac{b}{m} = Q_1\,;$$

then $$\frac{ab}{m} = mQQ_1 = aQ \text{ or } bQ_1\,;$$

therefore, $\frac{ab}{m}$ is a common multiple of $a$ and $b$.

But when $m$ is greatest, $\frac{ab}{m}$ is least; that is, $\frac{ab}{\text{G.C.M.}}$ is (*l.c.m.*) of $a$, $b$; or the product of $a$, $b$, divided by the greatest common measure, is their least common multiple.

If $m$ be (*l.c.m.*) of $a$, $b$, then the (*l.c.m.*) of $m$ and $c$ is also of $a$, $b$, $c$.

Thus, the (*l.c.m.*) of $4\,(1 - x)^2$, $8\,(1 - x)$, $8\,(1 + x)$, $4\,(1 + x^2)$ is evidently $8\,(1 - x)^2 (1 + x)(1 + x^2)$ or $8\,(1 - x)(1 - x^2)(1 + x^2)$, or $8\,(1 - x)(1 - x^4)$; that is, $8\,(1 - x - x^4 + x^5) =$ (*l.c.m.*)]

($q$). The terms which contain the same power of the letter according to which the arrangement is made, are to be considered as forming but one term.

Thus, in the fraction—

$$\frac{a^2(b^2 - c^2) - ab\,(2b^2 + bc - c^2) + b^3(b + c)}{a^3(b^2 + 2bc + c^2) - a^2b\,(2b^2 + 3bc + c^2) + ab^3(b + c)}$$

contemplating the co-efficients $(b + c)$, $(2b^2 + b$ $- c^2)$, $(b^2 - c^2)$, &c., it is at once seen that $(b + c$ is a common factor independent of $a$ . . . . suppressing it, we investigate the greatest divisor of

$$a^2 (b - c) - ab (2b - c) + b^3$$

and $$a^3 (b + c) - a^2 b (2b + c) + ab^3$$

which we find to be $a - b$. Hence that of the two terms of the fraction proposed is $a (b + c) - b (b + c)$; and it $\therefore$ reduces to

$$\frac{a (b - c) - b^2}{a^2 (b + c) - ab^2}$$

I subjoin two more examples of this kind of decomposition :—

(1.) $\dfrac{9x^3 + 53x^2 - 9x - 18,}{x^2 + 11x + 30}$ is evidently

$$= \frac{(x + 6) (9x^2 - x - 3)}{(x + 6) (x + 5)}$$

$$= \frac{9x^2 - x - 3}{x + 5}$$

(2.) $\dfrac{5a^3 - 10a^2 b + 5a\, b^2}{8a^3 - 8a^2\, b}$ may be thus decomposed :—

$$\frac{5a (a^2 - 2ab + b^2)}{8a^2 (a - b)} \text{, or, } \frac{5a (a - b)^2}{8a^2 (a - b)}$$

suppressing the common factor $a (a - b)$, the result is

$$\frac{5 (a - b)}{8a} .$$

($r$.) These, and similar reductions, which are of

the utmost value in practice, will be greatly facilitated by thoroughly studying the substance of the notes and remarks at pp. 124, 125, 130, 135, and the conversions at pp. 131, 132[1]. (ed. xi.)

A few more useful forms may be annexed to the margin of p. 130. Such as—

If $n$ be *even*, $a^n - b^n$ will be divisible by both $a - b$ and $a + b$.

If $n$ be *odd*, $a^n + b$ will be divisible by $a + b$, but not by $a - b$.

$a^m - a^n$ is always divisible by $a - 1$.

If $m - n$ be *even*, $a^m - a^n$ will be divisible by $a + 1$.

If $m-n$ be *odd*, $a \;\; + a^n$ will be divisible by $a+1$.

Many valuable examples are given in *Hind's* Algebra, and most of them with the *prospective* reference of which I have spoken, art. 9. Also in *Wright*, and in *Hirsch's* Collection, who says, "Upon this reduction the teacher has an opportunity of making several important remarks."

22. Circulating decimals and their most curious properties (note, p. 55), Duodecimals (p. 58), and Evolution (pp. 62—71), should be taught both with

1 The attention of the student should be carefully directed to all such operations, as, however easy, suggest, at the same time, a *principle* of simplification of frequent use. Such, for example, is No. 2, p. 134, (case 3), as may be shown by causing him to compare the work with the ordinary process.

a view to the facilities of practice and the comprehension of the *principles* upon which the practical rules are founded: and this, I repeat, is altogether within the reach of a cadet who has advanced to *Eliminations*, on admission.

(*a.*) Bearing this in mind, the methods of contracted multiplication and division should be thoroughly explained; as should also the contraction of the *square root* given at p. 63, with a simple demonstration of the *truth* of that method; such, for example, as is given by *Lacroix*, in his Algebra.

(*b.*) It might also sometimes be interesting to show in what cases the square root of a circulating decimal must, *of necessity*, be a circulating decimal. Such particulars having, independent of their immediate utility, an obvious tendency to excite, gratify, and keep alive, that spirit of active research which it is the great object of all these remarks and directions to produce and perpetuate.[1]

[1] It is well known that a whole number, having no integral root, has no decimal terminating root. But it is not to be *thence* inferred, that no fraction whatever can express the roots of such numbers, since there are fractions which cannot be expressed by any terminate decimal. And, if the decimal obtained in extracting the root of a number could, after any figure, become a circulating decimal, the value of the previous figures, together with the value of the fraction, from which such circulating decimal might arise, would be the exact root of that number. For example:—

(*c*). In teaching *Horner's method* for the cube root, which I regard as essential, or some analogous process, let the pupil be taught that it is NOT a method *of trial and error*, as some have inadvertently asserted, but a direct method *founded on*

if the figures obtained by extracting the square root of a certain number, were 3·412121212, &c. ad inf. $= 3{\cdot}4 + \frac{2}{165}$, the square root of *that number* would be $3{\cdot}4 + \frac{2}{165}$.

For, if possible, let P be a number, the figures of whose *m*th root at length circulate; and let $\frac{S}{D}$ be equal to the sum of the value of the previous figures, and the value of the circulating part; I say, that in that case, $\sqrt[m]{P} = \frac{S}{D}$, or $P = \frac{S^m}{D^m}$. For, if P be not equal to $\frac{S^m}{D^m}$, let it be equal to $\frac{S^m}{D^m} \pm d =$ $\frac{S^m \pm d\ D^m}{D^m}$ : then $\sqrt[m]{P} = \sqrt[m]{\frac{S^m \pm d\ D^m}{D^m}} =$ $\frac{\sqrt[m]{(S^m \pm d\ D^m)}}{D}$; that is, the figures obtained for $\sqrt[m]{P}$ agree throughout with the decimal fraction, by dividing the figures of $\sqrt[m]{}\ (S^m \pm d\ D\ )$ by D. But the figures of $\sqrt[m]{P}$, by hypothesis, agree with those obtained by dividing S by D. Therefore, the figures of $\sqrt[m]{(S^m \pm d\ D^m)}$ are the figures of S, or $\sqrt[m]{(S^m \pm d\ D^m)} = \sqrt[m]{S^m}$; which is absurd. Therefore, in that case $\sqrt[m]{P}$ would be equal to $\frac{S}{D}$: but because by hypothesis, P has no integral root, neither has it any fractional root, $\frac{S}{D}$. *Therefore the decimal obtained by the operation of extracting the root of a whole number, cannot circulate.*

*an approximation drawn from a binomial developement.*

The student should also be informed that, in cases where more figures are required in the root than the ordinary logarithm tables will supply, this method saves full *nine-tenths* of the work that would be required in Hutton's method (p. 68), and *forty-nine fiftieths* of the work required by the ordinary rule.

This may, farther, give the master an opportunity of enforcing the advantage of analytical talent in shortening and simplifying arithmetical operations, and proportionally *diminishing the risk of mistake in the work;* and a similar observation will apply to all Mr. Horner's beautiful simplifications, as well as to other matters brought forward in both volumes of the 11th edition of the Course.

23. Adhering to the plan all along recommended in these papers, of connecting, as far as possible, the subjects treated, both in arithmetic and algebra, let not only the fractions in both be thus connected, agreeably to the scientific views herein explained, but the progressions (pp. 79—85, 155—167), the Simple and Compound Interest (pp. 89—92, 258—264); the latter being, of necessity, deferred until logarithms are understood.

24. Fellowship and Loss-and-Gain should be treated as useful applications of the doctrine of

Proportion, elucidating, with especial care, the cases in which the gain or loss is estimated *per cent.*; and Single and Double Position, as obviously inferior and limited in application, but yet as manifestly subsidiary to Simple and Quadratic Equations.

25. Reverting now to considerations relating to the very commencement of algebra, let it be remarked, that clear views of the character and objects of this branch of mathematics, of the definitions, notation, and fundamental principles, are of the utmost importance.

I take it for granted (art. 11), that this constitutes an essential part of the revision to which the knowledge of every student is subjected on his entering an institution. The inquiry must go much farther than to the *recollection* of these matters. He must be taught what powers, as regards accuracy, facility, and generalization, are conferred by thorough familiarity with the use of algebraic symbols. He must be made to perceive and feel the *universality* of the algebraic language, as not only susceptible of immediate and safe application to the various topics of arithmetic, geometry, mechanics, hydrostatics, astronomy, &c., including extension, variations, motion, time, velocity, equilibrium, resistance, &c., but as evinced by the fact of its leading, in a comparatively short period, to

the most extraordinary improvements and discoveries: the calculus, which has verified the principles of gravitation, and enabled computers to predict various astronomical phenomena *to the beat of a clock's* pendulum, was but a century ago in its infancy!

(*a*.) Let the pupil, if he be advanced nearly to quadratic equations, be carried attentively over the definitions and general principles explained upon pages 104—115 of vol. i. edition 11.[1]

(*b*.) Let him attend carefully to the origin and advantages of adopting *letters* as symbols of number or quantity. (p. 106).

(*c*.)—To the precise force and application of the symbols that denote operations.

[1] There are three rules of a good definition, which should be attended to, not only in algebra and geometry, but throughout any scientific course. Science never proceeds so safely and successfully as when it proceeds with due regard to the principles of logic.

(Rule 1.) That a definition be adequate to the thing defined; that is, contain neither more nor less.

(2.) That it be proper to the thing defined, or distinguish it from all others.

(3.) That it be more clear and manifest than the thing defined.

N.B.—We simply treat here of the definition of the *thing*.

The definition of the *name* only explains the meaning of a *word*.

The too frequent defects of a definition are, that it neither agrees to the *whole* thing, nor to the *sole* thing.

(*d.*)—To the facilities supplied by the use, as far as possible, of *initial letters*; as *f*, for force; *m*, for momentum; *v*, for velocity; *t*, for time; *r*, rate of interest; *p*, principal, &c.; with the simple and advantageous exceptions, in purely algebraical problems, of restricting the initial letters to *known* quantities, and the final letters to *unknown* ones; and the infelicitous exception in the differential calculus, of employing the *d* to denote differential: an exception, however, which it is hoped will ere long be removed by the universal adoption of $\delta$, or of some other specific symbol, for differential.

(*e.*) Let the precise meanings of the technical words, as *co-efficient*, *exponent*, *residual*, *binomial*, *polynomial*, be fixed by such marked references to their etymology as shall preclude all mistake or ambiguity.

(*f.*) Let the analogies or relations suggested by the notation, be carefully specified, distinguished, and explained. As in $a^n$, . . . $a$, $a^3$, $a^2$, $a$, $a^{\frac{1}{2}}$, $a^{\frac{1}{3}}$, $a^{\frac{1}{4}}$, $a^{\frac{1}{2}}$. $a^n$ . . . . $a^4$, $a^3$, $a^2$, $a$, $a^{-1}$, $a^{-2}$, $a^{-3}$, $a^{-4}$, $a^{-n}$, &c., &c., &c., (pp. 109—111.)

(*g.*) Let the cases in which a vinculum is necessary, or unnecessary, be carefully discriminated; and the advantages and disadvantages of the different characters employed for the purpose be as carefully marked.

(*h.*) Let the *power* of algebraical language, as a correct vehicle of thought, as an instrument to express the utmost variety of conditions, be most sedulously and distinctively impressed. (See pp. 112—114.)

It is one thing to learn to express the conditions or terms of a question, or proposition, intended for investigation, and quite another to conduct the investigation itself. The *former* is what is here meant; though *Lacroix*, and other excellent writers on algebra, actually commence the subject with the solution of some equations,—a method which, I am persuaded, has the most decided advantages, although we are not yet *quite* prepared to adopt it universally in this country.

(*i.*) Let the student be thoroughly versed in the art of determining the numerical values of different algebraical expressions, the respective value of each letter being assigned. (See p. 115.)

[The last two directions relate to matters of such extreme importance, that, I doubt not, it will often be found expedient to give the student a greater variety of exercises than the "Course" exhibits; especially when it is borne in mind that students are frequently found extremely defective with regard to both these particulars. Several good examples for practice are given by Professor *Gill*,

of the Flushing College, in his *Mathematical Miscellany*.]

(26.) In addition to such of the preceding directions, (art. 20, *a*, *b*, *c*, &c., 21, *a*, *b*, *c*, &c.) as relate to algebraic principles and operations, I now present a few more, as the subjects occur in the order of the printed Course.

(*a*.) Let the note, p. 116 (*Addition*), be attentively explained.

(*b*.) With regard to the note, p. 121 (*Multiplication*), see the foregoing art., 20 *n*.

(*c*.) Let Horner's improved methods of multiplication and division, by the detached co-efficients, and his beautiful method of synthetic *division*, (pp. 123, 129, 130), be explained and enforced, both as relates to theory and practice, with a view to their use in relation to series, and in the solution of equations (pp. 210, 226, 228), and such as

$$x^2 - 25x = 88$$
$$4x^2 - 20x = 352,$$

with a caution and skill proportionate to their analytical value.

Synthetic division, besides its immediate use, is, as the student should be apprized, of great value in obtaining factors, preparatory to the integration of finite differences, in the method of combinations, the construction of a recurring series, the treat-

ment of reciprocal equations, and many other interesting researches.

(*d.*) At p. 145 (*Evolution*), let the characteristic difference in the two cases of $(-a-b)$, raised to integral powers, with *odd* and *even* exponents, be made clear by at least two additional examples. Let the curious property in the note p. 152, conneeting the sum of the co-efficients with the powers of 2, be demonstrated; and let not the practical remark at the foot of p. 143, (*Evolution*), as to *performing the division first*, be disregarded.

(*e.*) Let the whole doctrine of surds be taught with the *prospective* reference, to which I have before adverted (art. 9), as well as to the most useful and scientific decompositions of the factors.

(*f.*) Let the student be made to keep in mind the practical advantages of removing the surd *from the denominator* of a numerical fractional expression; and of such simplifying reductions as are explained in the notes, pp. 149, 150, 153. ex. 11.

(*g.*) In explaining and demonstrating the note, p. 153, relating to a binomial or residual surd, let the pupil be shewn how it depends upon this property, that the square root of a rational quantity *cannot* be partially rational and partly a quadratic surd.

(*h.*) And from hence let it be shewn, (again with the prospective reference which I consider as so highly important in exciting a thirst for knowledge,

and stimulating a love of inquiry), that, with regard to the elimination of unknown quantities, there is an exception to the rule that there must be as many *independent* equations as there are unknown quantities in the case,

$$x \pm \sqrt{y} = a \pm \sqrt{b}; \quad \text{and why?}$$

And, with reference to the whole business of equations, from simple equations onward, let a marked and decided care be taken that with the pupil it should never be permitted to slide into a mere matter of mechanism, but preserved as one of mind also, involving the progress of method and reasoning: and, with a view to this, let the master especially enforce the directions in the valuable note at p. 178, at the very commencement of the rules of Simple Equations.

(*i.*) Let the student, on being taught the two first as well as the third problem in *Series* (pp. 168—176), be instructed, not merely in particular examples, but shewn how, in many cases, one form includes several particular series, and thus led to deduce a few of the generalizations which they so obviously indicate. Let him here, also, learn to discriminate between the two uses of the mark (=); 1st, that of a sign of *absolute equality*, as when we read, that $11 = 5 + 9 - 3$

$$13^2 = 5^2 + 12^2$$

$$(m^2 + n^2)^2 = (m^2 - n^2)^2 + (2mn)^2.$$

2ndly, that of a symbol or indication of *a limit*; as in the cases,

$$\frac{1}{1-\frac{1}{2}} = 1 + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \tfrac{1}{16} + \text{\&c.},$$

$$\frac{a}{1-r} = a + ar + ar^2 + ar^3 + ar^4 + \text{\&c.},$$

the former approximating more and more to a limit, properly speaking, the farther it is continued: the latter doing so when $r$ is less than unity.

(*k.*) Here, also, he should be told, as the terms may also occur without explication, that *convergency* is continued approximation to limit; *divergency*, the contrary; and that a slight change in the relations of the quantities employed, may change a *converging* into a *diverging* series; and the contrary.

(*l.*) Among the examples of series whose peculiarities may be advantageously explained to the student, take the following:—

$$(1.)\quad \frac{1}{1+1} = 1 - 1 + 1 - 1 + 1 - 1 + \frac{1}{1+1}$$

$$- = 1 - 1 + 1 - 1 + 1 - 1 + 1 - \frac{1}{1+1}$$

Here, if the series terminate at an even number of terms, those with + and those with − destroy each other; and the limiting value of the series is $\frac{1}{1+1} = \frac{1}{2}$, the remaining fraction.

But, if the series stop at an *odd* number of terms, we have $1 - \frac{1}{1+1} = 1 - \frac{1}{2} = \frac{1}{2}$, for the limiting value.

(2.) $\frac{1}{1-a} = 1 + a + a^2 + a^3 + a^4 + \frac{a^5}{1-a}$

If $a = 1$, then,

$$\frac{1}{1-1} = 1 + 1 + 1 + 1 + 1 + 1 + \&c. = \frac{1}{0} = \infty$$

If $a = 2$, then,

$$\frac{1}{1-2} = 1 + 2 + 4 + 8 + 16 + 32 + \frac{64}{1-2}$$
$$= -1.$$

If $a = 3$,

$$\frac{1}{1-3} = 1 + 3 + 9 + 27 + \frac{81}{1-3} = -\frac{1}{2}$$

[The two latter exemplifications are useful, as showing how a series of *positive* numbers is derived from an expression essentially *negative*; and how the paradox is removed.]

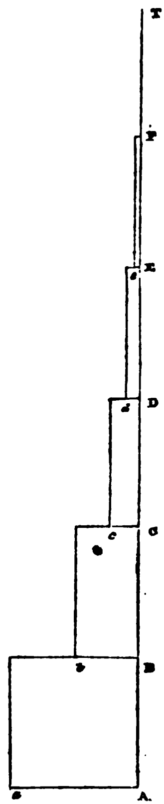
If $a = \frac{1}{2}$,

$$\frac{1}{1-\frac{1}{2}} = 1 + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \tfrac{1}{16} + \tfrac{1}{32} + \&c.,$$

2 the limit.

3. It may be well to give the student a geometrical illustration of this case of the general form, to the following effect:—

Let AT be an indefinite right line, on which set off the equal parts, AB, BC, CD, DE, &c. On AB make a square; on BC a right-angled parallelogram, whose breadth, B*b*, is half the breadth A*a* of the square; on CD, another rectangular parallelogram, whose breadth, C*c*, shall be half B*b*, and so on. It is evident that the right-lined figure composed of all these rectangles, though infinite in extent from A towards T, is finite in area, being equal to *twice* the square upon AB; or the aggregate of all the areas approximates to that, as a limit.

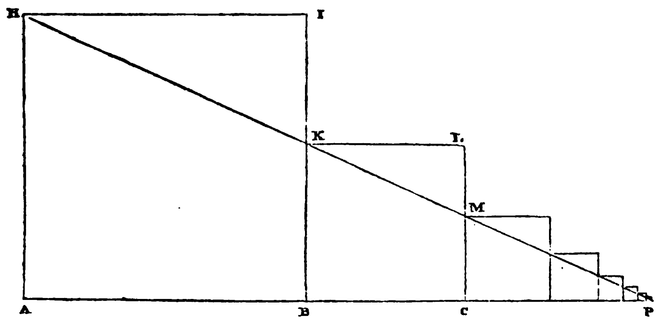



(4.) The demonstration of No. 64 of the exercises in plane geometry (p. 334), will supply another instructive illustration of the same principles.

The proposition is this:—The sum of a descending infinite series, such as $a + b + \frac{b^2}{a} + \frac{b^3}{a^2} +$ &c., is well known to be expressed by $\frac{a^2}{a-b}$, the square of the first term divided by the

difference of the first and second. Demonstrate this upon geometrical principles.

In the indefinite right line, ABC, &c., let AB represent the first term ($a$), and BC the second




term ($b$) in the series. Upon this right line draw the squares ABIH and BCLK; through the points H and K let a right line be produced, and it will cut the indefinite right line ABC, &c., in the point P. Now, the line AP will represent the sum or aggregate of the infinite series in the proposition. For the squares may be supposed to be continued on in *infinitum* as to number, and their sides must inevitably bear the same proportion with the terms in the series; and that the sides of the squares in continuation cannot go beyond the point P, is manifest, since the side of every new square to be raised will be to the remainder of the line AP, as HA to AP, or KB

to BP, or MC to CP, &c. Therefore, to ascertain the length of AP, or the sum of the infinite series, we have, from the similar triangles, KIH, HAP, KI : IH : : HA : AP; that is, $a - b$ : $a$ : : $a$ : $\frac{a^2}{a - b} = \text{AP}$. Q.E.D.

(*m.*) Let the correction of the errata at p. 175 be made an instructive exercise for the student. Let him be taught the use of this prob. iii. in tolerable detail, showing him on what the convergency depends, and explaining in what manner, as pointed out in pp. 175, 176, this valuable method may be made *universally* applicable, and not be limited in its application, as some have either ignorantly or inadvertently asserted.

(*n.*) As to the higher portion of Infinite *Series* (pp. 264—284), it will, in my judgment, be usually expedient to defer it, to be taught in the 2nd or 1st Halls of Study, and there for the preceptor to point out the occasional prospective references to matters connected with the Differential and Integral Calculus. But the principles and use of the method of *Indeterminate Co-efficients*, with the investigations of the binomial and exponential theorems (pp. 238—244), may advantageously precede and be connected with the subject of Logarithms.

27. Let the doctrine of Simple and Quadratic Equations be taught, not merely in order to evince

their power as instruments of investigation, and to illustrate and impress the various ingenious, and sometimes beautiful expedients, by which their solution is effected; but also with a prospective reference to the solution of equations in general. Showing, for example,

(*a*.) How it happens that in such questions as No. 13, p. 182, before the *simultaneous equations*, there is another value of $x$, besides the printed answer; and why it at first escapes detection?

(*b*.) How, by the rules of elimination, the order of an equation may often be depressed. As at p. 200, rule iv. ed. 11.

(*c*.) How it happens that a quadratic *cannot* have more than two roots.

(*d*.) How a quadratic, whose term with the square has a co-efficient, as $7x^2 + 4x = 36$, may be solved, without involving fractions, by the *Indian* method: and how that method may be just as advantageously applied to cases in which the co-efficient of $x^2$ is 1, and that of $x$, any odd number; thus making the method universal. Which is clear manifestly from this:—Where $ax^2 + bx \doteq c$, is an equation of which $b$ is always odd, then

$$4a^2 x^2 + 4abx = 4ac, \text{ add } b^2$$

$$4a^2 x^2 + 4abx + b^2 = 4ac + b^2,$$

extract, then, $2ax + b = \pm \sqrt{4ac + b^2}$

hence, $$x = \frac{-b \pm \sqrt{4ac + b^2}}{2a}.$$

(*e.*) How, finding all the roots of $x^3 - a = o$, $x^4 - a = o$, $x^5 - a = o$, &c., tends to facilitate the general solution of equations; and so with regard to many other particulars not here specified.

(*f.*) How the second term of a quadratic equation, $x^2 \pm px = q$, may be taken away, and thus reduced to a pure quadratic, after a manner analogous to that in which the second term in cubics is taken away.

(*g.*) How it happens that such equations as

$$8x - 2x^2 + 4\sqrt{2x^2 - 8x + 15} = 3$$

or
$$\frac{x + \sqrt{x^2 - 4}}{x - \sqrt{x^2 - 4}} = (2x - 3)^2$$

though apparently quadratics, are, in fact, biquadratics, and have each four roots.

28. It may also subserve the same purpose, as well as again illustrate the doctrine of limits, in another view of it, if the student be briefly taught the management of inequalities or *inequations.* [Not inserted in the Course for want of room.]

Take this example:

The double of a number diminished by 5 is greater than 25; and triple the number diminished by 7, is less than double the number increased by 13. Required a number which will satisfy the conditions.

By the quest. $2x - 5 > 25$
and $3x - 7 < 2x + 13$
Hence, $2x > 30 \therefore x > 15$
also, $3x - 2x < 13 + 7 \therefore x < 20$

$\therefore$ 15 and 20 are the limits; and any number, integral or fractional, between 15 and 20, will answer the conditions. [More of this class of questions are inserted in Bourdon's Algebra.]

29. Let the nature and operations, by *logarithms*, be explained, as other matters, with specific reference both to theory and practice. Comprehending thus in the lessons:—

(*a.*) The fundamental principles.

(*b.*) The characteristic differences of Napier's and Briggs's logarithms.

(*c.*) The investigation of series for the computation of logarithmic tables.

(*d.*) The ready employment of tables carried to 5, 6, 7, or more places of decimals; as hinted at p. 253, vol. i.

(*e.*) The practice of the several rules, and their use in the ordinary as well as the trigonometrical proportions, and in compound interest and annuities.

(*f.*) The solution of exponential equations.

(*g.*) The developement of one or more scientific reasons, why the *difference* between one log. and

the succeeding log. is greater near the *beginning* than near the *end* of the table. As, for example, greater between log. of 100 and of 101, than between the logs. of 1000 and 1001, and still more than between those of 9000 and of 9001.

(*h.*) The rule for ensuring accuracy in the results of Evolution by logs., when the number whose root is extracted is *altogether decimal* (note, p. 254), and the theoretical reason of that rule.

30. With regard to EQUATIONS in general, although what is introduced between p. 208 and p. 239 is meant as a synopsis of what is most interesting and instructive in this department of analytical inquiry; yet even this portion must be studied with an especial view to the distinction between what is rudimental, subsidiary, and of restricted application, and what is more refined in its analysis, as well as universal in its use. Thus—

(*a.*) With regard to Cardan's rule, let it be explained, first, with a view to the preparation of the equation by taking away the second term, (a particular case of a general rule); then let the expedient for solving the transformed cubic be elucidated by examples, and the ingenuity of the peculiar assumption

$$\text{of} \quad x + y = z$$
$$\text{and } 3xy = -a$$

be shown, by which the solution is effected.

Then let it be shewn how the sign of $a$, in the transformed equation $z^3 \pm az = \pm b$, determines the nature of the roots as to real and imaginary; that of $b$, the affection of the roots as to positive and negative; how, also, it is that Cardan's rule sometimes gives the *greatest*, sometimes the *middle*, sometimes the *least*, and not always the *greatest* root, as has been frequently thought: then, let the cases be adequately traced in which the solution by Cardan's rule takes an imaginary form (p. 212); and let the use of this rule, in determining under what conditions amongst the data of a problem a cubic equation, in its *literal* form, can have only one root; and under what conditions it may have three roots; such being now, in fact, the principal use of Cardan's method.

(*b*.) Let Simpson's and other rules for biquadratics, be slightly glanced at, as supplying an interesting disideratum in the history of equations; but not now dwelt upon.

(*c*.) Let the method by *Trial and Error* be thoroughly illustrated and enforced, both in theory and practice, and the cases fully explained (see pp. 215, 256), in which this method may *still* be advantageously employed, notwithstanding the decided improvements made by Horner and Sturm.

(*d.*) Let this rule by trial-and-error be farther exemplified, subsequently, in its connection with trigonometry and mensuration. As, for example,

(1.) In determining the arc of a circular sector whose area is bisected by its chord.

[Here, if $\phi$ denoted the arc, we should have

$$\frac{\pi r^2 \phi}{360^\circ} = r^2 \sin \phi$$

$$\text{or} \quad \frac{\pi \phi}{360^\circ} = \sin \phi$$

assuming $\phi = 90^\circ$ and $120^\circ$, and working by logs. the first operation would give $\phi = 108^\circ\ 30'$ nearly.

Then assuming $\phi = 108^\circ\ 30'$ and $108^\circ\ 35'$, the next approximation would be $108^\circ\ 36'\ 13''$.

A third double assumption of $108^\circ\ 36'\ 13''$ and $108^\circ\ 36'\ 14''$ would give the arc still nearer, if required: and a fourth would determine $\phi = 108^\circ\ 36'\ 13''\ 45'''\ 27''''$.

This process is founded on the well-known property of trigonometrical lines and their logarithms, that in small alterations of the angles, or arcs, their differences are *nearly* as the differences of the angles themselves; a supposition which is the more correct the smaller the changes are with respect to the angle.]

(2.) A second example of the use of this method might be found in solving the inquiry, probably

paradoxical at first sight: Find the first four in the series of circular arcs which are equal in length to their tangents.

[Euler's *Analysis Infinitorum*, cap. 22, would furnish other exercises for the leisure of students of active and persevering research. In ordinary cases the above, or perhaps the first of them, may suffice.]

(*e.*) Let the student be conducted with especial care over the chapter on the composition and properties of algebraic equations, until he thoroughly comprehend the manner of tracing the object and import of the *permanencies* and *variations* of the signs; the manner of transforming equations into others whose roots shall have specified relations to the roots of a given equation; the manner of proceeding with equations known to have *equal* roots; the relative character and value of the criteria of *Maclaurin*, *Bret*, *De Gua*, *Budan*, and *Fourier*, in reference to the *limits* of the roots of equations; then the method of finding the initial values, or first figures, of each of the roots; which should be carefully explained and enforced, as peculiarly useful in determining an essential point, in every method; and lastly, Mr. Horner's elegant and strictly scientific, as well as highly practical method of resolving *algebraic* equations of all degrees; devoting full time and attention to the elucidation of his theoretical processes. That method, in conjunction

with the researches of Sturm, which Mr. Spiller has recentlypresented to the British public, make a nearer approach to the complete solution of Equations, than all the labours of mathematicians during the eighteenth century.[1] See, also, Young on Equations.

31. Although the algebraical and geometrical branches of study should be carried on simultaneously by an appropriate adjustment of time, as already observed (art. 11), yet in these directions it is necessary to speak of them separately.

32. At the very outset of geometry, let the student be shewn the characteristic nature of geometrical demonstration; in which the agreement of the remote ideas in a specific case, or proposition, is made manifest by the agreement of a train of intermediate ideas, each agreeing with the next to it, and all *fixed* by definitions, as well as connected with some manifestly sound principle, either axiomatic or previously demonstrated: let him be shown, also, why this kind of demonstration is far more convincing than any other; and what is its

[1] I need not hesitate to refer here to Davies's *Solutions to the Questions in Hutton's Course of Mathematics.* It presents, in many respects, a very lucid commentary on the subjects, as well as a collection of neat and valuable solutions. See, for example, the copious information on Equations generally, and on Series, &c., from p. 207 to p. 309, and in many other places.

advantage in giving firmness to the several links, as well as in marking the concatenation, and in giving stability and force to the progression of the reasoning in the entire course. (See arts. 3 and 18, Treatises on Geometry.)

33. Let him, also, be taught to distinguish between the compact and concise announcements and proofs of geometrical propositions which are written and read mainly for *practical* purposes, and those which have a higher, and often an exclusively intellectual object; and of these let him distinguish between such a system as Euclid's Elements, in which geometrical truths are presented, and each demonstrated, in an orderly logical series; and those works in which the *art* or process of demonstration is exhibited and enforced: to distinguish, in other words, between the adequate recollection and clear comprehension of the successive propositions in a series, tracing their mutual dependence, and intellectual beauty; and that exhibition of geometrical research which converts it into an instrument of investigation; by means of which, instead of always *receiving* propositions demonstrated by others, and nothing more, the student is put in the way of *using his own instruments, and investigating for himself.*[1]

[1] In truth, notwithstanding the wonderful fecundity of the modern Analysis, Geometry may still put in a strong claim to the

34. To this end, let him early be made to comprehend the technical distinction of *analysis* and

exhibition of the best logic; since, while it teaches us to reason by the fittest examples, and employs the understanding in demonstrating the properties of invisible quantities, it represents those quantities visibly to the imagination, which it thus confirms, and as wonderfully generalizes the attributes of the peculiar magnitudes with which it is conversant.

Let us, for a moment, speak of the general rules of method, and then describe briefly the method of geometers.

In treating completely and adequately of any subject, (1.) There should be nothing either wanting or superfluous; (2.) The parts must agree and adhere; (3.) There should be nothing foreign to the subject; (4.) The parts should be connected by apt transitions; (5.) That should precede in teaching, which is necessary towards the mental comprehension of what follows.

Now good mathematicians use this method:—

(1.) They define their terms, and use them always in the same sense.

(2.) To their definitions, they join axioms, or self-evident truths, which they see will be useful in the progress of their reasoning.

(3.) They next add postulates, which relate to practice. These are also self-evident, and therefore they may require them to be granted without proof.

(4.) In the next place, they demonstrate propositions, as far as possible, affirmatively; and make it an invariable rule to prove every thing from what has been already granted or demonstrated.

Hence, the evidence compels our assent: and hence we generally—universally, indeed, if the process be carefully conducted—avoid all danger of confounding the figure or diagram with the thing represented, or of assenting to obscure notions as though they were clear,—the fruitful source of all our errors.

*ynthesis.* If we trace a magnitude or quantity from its *known* property, (and we can only reason from what we know), the method is that of *analysis;* but if we trace a *property* from the kind of quantity or magnitude before us, the method is that of *synthesis;* and a mathematical demonstration of either kind proves the connection between any quantity or magnitude and the property ascribed to it. (pp. 2, 3, vol. i.)

It is often highly instructive to trace both the analysis and the synthesis of a proposition. The student should, after he has studied Euclid's first five books—if not, indeed, before—be conducted carefully over a few select propositions from pp. 371—376, vol. i. of Hutton, and from Euclid's admirable book of DATA, in R. Simson's edition.

35. Let the student be at once told that Euclid's Elements are adopted as the text-book in plane geometry, because (notwithstanding several defects which should be carefully pointed out) of the conviction flowing from the separate demonstrations, as well as of the beauty and force of the logical concatenation; but, that Hutton's geometry of solids, and of intersecting planes, is often adopted in preference to Euclid's 11th and 12th books, not as being preferable in point of accuracy, but as presenting analogous truths in a more compendious

form, and as being greatly improved in the last edition.

36. In commencing a book of the Elements, as the first for example, let the separate character and object of *definitions*, *postulates*, and *axioms*, be clearly explained; let such of them as are unintelligible, defective, or redundant, be pointed out and amended (not with a view to manifest or encourage a spirit of hypercriticism, but to check the too frequent habit in youth of taking all, or nearly all, for granted, and to foster in them a determination to examine every thing important with the closest scrutiny): *occasional* illustrations of Simson's views, and of the master's, should be selected from the notes in the volume, or otherwise presented. What are here given may suffice, perhaps, as specimens.

(*a.*) Thus, let Simson's note to def. 1, book i., be read and carefully explained orally, in illustration of some of the earlier definitions.

(*b.*) With regard to def. 8, let it be observed, that it is not, in fact, a definition, but a theorem; the requisite addition would be simply this:—"For, if not, let any part of the straight line lie *out* of the superficies, the superficies is then uneven; that is, it is not plane, which is contrary to the supposition."

(*c.*) Let it be remarked, also, that it is as de-

sirable to have right angle and perpendicular (def. 10), in two distinct definitions, as a circle and its diameter (def. 15, 17). It ought, farther, to be proved, that all right angles are equal to one another.

(*d.*) Also, that def. 13 is needless, because a *boundary* is a *term*, and the phrase explains itself.

(*e.*) Let the imperfections in defs. 4, 13, 30, be pointed out, and removed.

(*f.*) Let it be shewn that def. 19 of this book is the same as def. 6 of the third; and that what is called an *oblong* (def. 31) is called a *rectangle* (def. 1, book ii.), and that the latter is the best designation.

(*g.*) Let some of the defs. be *fixed* more firmly in the memory, by means of classification. As, for example, that triangles are divided into three varieties with respect to their *sides*, viz., *equilateral*, *isosceles*, and *scalene ;* and to three with respect to their *angles*, viz., *right-angled*, *acute-angled*, and *obtuse-angled*.

(*h.*) Let the difference between the phrases, *makes an angle*, and, *makes angles*, be explained.

(*i.*) Let not the difficulty relative to def. 35, and axiom 12, be evaded; yet let it be assigned as a reason for not going through the whole of Simson's elaborate note on prop. i. 29, and ax. 12, that in

the 5th prop. of that note he takes for granted a particular case of the general proposition intended to be established.

(*k.*) Let it be shown that the right-angles mentioned in ax. 11, are not restricted to the case of def. 10, but mean *remote* as well as *adjacent* right-angles. See Cor. to prop. 11.

37. On entering upon the enunciation of propositions, let Euclid's meaning, and some of his uses of the word *given*, be elucidated by referring to the definitions in the book of data.

38. Let it be observed in reference to book i.

(*a.*) That props. 2 and 3 have several cases; though they need not be dwelt upon.

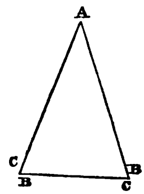
(*b.*) That prop. 4 involves new considerations, those of motion and supraposition: how can a plane triangle, according to the definition, be moved and laid upon another? Or, can this consideration be avoided by another demonstration? Can the difficulty be surmounted by meditating upon a prism, with two equal triangular bases, or ends? These are equal *surfaces*, which may be brought into juxtaposition, by cutting the prism into two.

(*c.*) Prop. 5 may be made plainer, by dissecting and separating the diagram into 3 or 5; or the demonstration may be otherwise conducted.

The proposition, as to the first part, may be

proved by means of the 4th and the 13th, which last might have been demonstrated before the 5th. Or, drawing AI to bisect the angle BAC, then BA, AI, and angle BAI, are respectively equal to CA, AI, and angle CAI; therefore, B = C, (th. 4), and CBD = BCE.

To prove that the angles above the base are equal, *Pappus* proceeded thus:




Let ABC, and ACB, be two isosceles triangles, whose sides are respectively equal. Conceive the two triangles to be coincident, or to be as one. (The imagination will seize *Pappus's* thought at once, by conceiving the triangle ABC to be drawn on one face of a piece of transparent paper, and ACB to be seen on the other side of the transparent paper.) Then, because in the two triangles ABC, and ACB, AB is equal to AC, and AC to AB; therefore, two sides of the one are equal to two sides of the other, each to each; and the angle BAC is equal to the angle CAB, being in fact one and the same angle; wherefore, by prop. 4, the base CB is equal to the base BC, the triangle BCA = CAB, &c., as in the proposition.

(*d.*) Prop. 6 has two cases in the demonstration, and so have several others; as, for example,

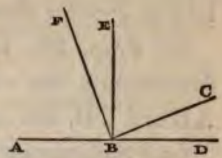
propositions 39 and 40; these should be *noticed*, and in one or two cases dwelt upon.

(*e*.) The demonstration of prop. 8 does not prove it, for the angles do not *fill* the same space. The defect arises from Dr. Smith's interpolation in the 8th axiom, *i. e.* "that is, which exactly fill the same space." How, in the demonstration, is the 8th axiom employed? surely not in reference to *filling* the same space; angles do not *bound* space, and angles may be *equal*, without an entire coincidence of their limiting lines. Farther, what is space, and where defined?

(*f*.) The construction of prop. 9 wants an addition to the enunciation, to remove ambiguity. After, "join DE, and upon it describe," add, "on the side *towards* B and C."

(*g*.) Prop. 11, the case, when the given point is at an extremity, as A, of the line, should also be constructed and demonstrated, by principles established previously to that proposition.

Also, since it is by no means clear, whether axiom 11 is meant to refer to angles adjacent to the same perpendicular, (which are equal, def. 10,) or to those which are formed by different perpendiculars, with different right lines; the corollary to this 11th proposition, should be differently made out, thus:




For, if it be possible, let the lines BC, and BD, have a common segment AB, and let BE be perpendicular to ABC; if it be also perpendicular to ABD, then the angles CBE, DBE, are equal (by axiom 11), which is absurd; but if not, let BF be perpendicular to ABD; then the angles ABF, ABE, are equal, which also is absurd.

(*h.*) Prop. 20 admits of another and simpler demonstration, upon Euclid's previous principles.

There is also a looseness in the phraseology of the construction, which often occurs in Simson's edition, and should as often be corrected. Instead of "produce BA to the point D, and make AD equal to AC," it should be, "produce BA until its prolongation AD is equal to AC," &c.

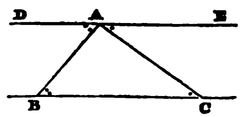
(*i.*) Prop. 26, the phrases, "the base GC is equal to the base DF,"—"the base AH is equal to the base DF," are needless: a similar redundancy occurs in the demonstration of book i. 47, and in various other places.

(*k.*) With regard to prop. 27, it may be asked, where are alternate angles defined? Observe also, that the lines AB, and CD, must be in the same plane; otherwise, the alternate angles might be

equal, and the lines might, or might not, be parallel; as *Proclus* shows.

(*l.*) Prop. 30: let the student be required to demonstrate the case, when EF is not *between* AB and CD, but *exterior* to one of them.

(*m.*) Sometimes require the student to vary the demonstration of prop. 32, as suggested by this diagram.

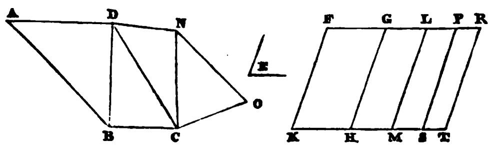



(*n.*) Prop. 40: notice that the bases BC, EF, are supposed to be parts of one right line.

(*o.*) Prop. 44: if BE partly coincided with BA, and Euclid does not prohibit such a position, the construction would not solve the problem. Farther, in tacitly supposing the parallelogram, made equal to the triangle C, to be transferred to AB, Euclid assumes a new postulate.

After, "the same straight line with AB," add, "that is, in AB produced."

(*p.*) Prop. 45: the enunciation extends to any multilateral figure; the demonstration applies to the particular case of a *quadrilateral.* How may it be extended?




Let ABCOND be the rectilinear figure; join DB, DC, CN. Thus, having made ▭ KEML = ABCD, as in the prop., apply LS = DCN, to the straight line LM, having an ∠ LMS = E; then it may be proved, as before, that FL and LP are in the same straight line, as are also KM and MS; also, that PS is parallel to FK, and, therefore, that FKSP is a ▭ = ABCND. Applying, as before, another ▭ PT = NCO, having angle PST = E, to the right line PS; it may again be proved, that FS is a ▭ = to ABCOND, and an ∠ FKT = E. And so on, for any number of sides.

(*q*.) In such propositions as the 16th and 47th, where part of the demonstration is omitted, but its necessity suggested by the phrase, "in the same manner;" let the student be required to supply that part of the demonstration.

(*r*.) Prop. 48: It is assumed in the demonstration, without proof, that if DA = AB, then $DA^2 = AB^2$; and *vice versâ*. Let both be demonstrated in a separate exercise.

**39.** After the first book has been gone through,

let the student be subjected to the process of revision.

(*a.*) By his expressing in writing, (probably entering it immediately above the definitions,) a very brief summary, in three or four lines only, of the principal contents of the book.

(*b.*) By demonstrating, *vivâ voce*, and promptly, any specified propositions contained in the book itself. This may most readily be done, by requiring demonstrations of the propositions referred to, in three or four of the later propositions; as, for example, propositions 4, 14, 31, 41, referred to in proposition 47; and propositions 8 and 11, cited in prop. 48; but it is evident, that the propositions, whose demonstrations shall thus be called for, must be much and often varied.

(*c.*) By proposing some suitable exercises, whose constructions and proof may be drawn from propositions in the first book (see p. 331, vol. i.)

(*d.*) By showing that certain propositions are the converse of certain other propositions, that is, as logicians express themselves, the subject of one is the predicate of the other, and *vice versâ;* and that, contrary to what Emerson asserts, it is often, *not* "an idle waste of time to demonstrate the converse of a proposition."

Exemplify this in a few cases; previously remarking, that there are different kinds of converse

propositions, of which the first and chief kind is, when that which was the thing *supposed* in the former proposition, is the *conclusion* of the converse and second proposition; and contrariwise, that which was *concluded* in the first, is the thing supposed in the second. Thus, in theor. 2, or prop. 5, there are *two* conclusions; the first, that the two angles *at* the base are equal; the second, that the two angles *under* the base are equal; while theor. 3, or prop. 6, is the converse of prop. 5, simply, as to the *first* conclusion; the second conclusion has not its converse established in the ordinary demonstration of prop. 6, but requires an obvious modification of the former proof to comprise it. So again, as *Billingsley* long since remarked, (A. D. 1570), though theor. 5, is the converse of theor. 4, it is not the principal kind of conversion; "for it turneth not the *whole* supposition into the conclusion, and the *whole* conclusion into the supposition."

(*d.* 2.) The converse of prop. 15 is, if four right lines being drawn from one point make four angles, of which the two opposite angles are equal; the two opposite lines shall constitute one right line. This proposition is demonstrated by *Proclus* and *Pelitarius*.

(*d.* 3.) And again; proposition 34 includes three distinct properties, involving the equality of the

opposite sides, the equality of the angles, and the truth that either diameter divides the parallelogram into two equal parts. *Proclus*, *Flussas*, *Pelitarius*, have made out the converse of each of these cases.

(*d*. 4.) These, and many other cases of converse propositions, often produce practical results of great value; and are here simply alluded to, though briefly, because they serve to convince a student of acumen and research, who will not satisfy himself with gliding over the surface of things; how admirable is the fecundity of even this region of truth, and what is the corresponding reward of those who do not rest satisfied with a hasty glance at what lies before them.

Let it be observed, also, that *converse* and *contrary* propositions are not to be confounded. Two propositions are *contrary* to one another, when one affirms what the other denies, or denies what it affirms. Thus, if it be affirmed, that three and four *are* seven, it is a contrary proposition that three and four *are eight*.

(*e*.) Once more, with respect to the first book of the elements, notice that it, like most of those which follow, supplies various examples of the fecundity to which allusion has just been made, that almost creative power of generalizing, which so frequently appears in the researches of geometers. For example, in tracing the properties of triangles,

with regard to their sides and angles, taking of the six data *three and three* together, there will be found twenty combinations; of which, prop. 4 involves *three*, while prop. 8 involves only *one*, and prop. 26 *nine;* the remaining *seven* have not fallen under the contemplation of Euclid. The synopses of the data of triangles by *Lawson*, *Leybourn*, and *Farey*, have added very greatly to the rich varieties of this kind of classification.

(A similar course, of greater or less extent, according to circumstances, may be adopted at the end of each book of the elements.)

These and other matters, which might have been specified, but which will occur to the experienced teacher, are not verbal niceties, but are essential to scientific accuracy and perspicuity. Regarding them or neglecting them constitutes a main distinction between logical, scientific instruction, and perfunctory lessons in which memory is substituted for the active investigating powers of intellect.

40. The word *rectangle* often occurs in the enunciations and demonstrations of the second book; but in several editions of Simson, that word is not defined; when such is the case, let definition 1 be amended to run thus:

"Every right-angled parallelogram is called *a rectangle*, and is said to be contained by any two

of the straight lines which contain one of the right angles."

(*a.*) In demonstrating the propositions of this book, students who have been allowed to employ Williams's symbolical Euclid, instead of Simson's, are very apt to use the algebraical terms *plus*, *into*, &c., instead of the corresponding geometrical phrases *together with*, *rectangle under*, or *rectangle contained by;* let them be especially guarded against this loose, and ungeometrical mode of expression, as it is calculated to fill the mind with vague ideas.

(*b.*) In the demonstration of prop. 4, instead of the sentence "for CG is parallel to BK," &c.—— down to "and CGKB is rectangular," let this be substituted; "for KBC is a right angle by construction, therefore, CGKB is rectangular (i. 46, cor.)"

Several other propositions in this book may be demonstrated more simply, still retaining Euclid's manner. I shall briefly indicate the demonstrations of 2 or 3 of them, such as may be written on the margin of the book.

(*c.*) Proposition 7.

Because $AB^2 = AC^2 + CB^2 + 2AC.CB$ (ii. 4),

and $CB^2 = CB^2$

$\therefore AB^2 + CB^2 = AC^2 + 2CB^2 + 2AC.CB.$

But $CB^2 + AC.CB = AB.BC$ (ii. 3.)

$\therefore 2CB^2 + 2AC.CB = 2AB.BC$

$\therefore AB^2 + CB^2 = 2AB.BC + AC^2$.

(*d.*) Supposing it to have been proved in the demonstration of i. 48, that when $AE = BD$, then $AE^2 = BD^2$, then prop. 10 may be demonstrated thus:

Produce CA to E, making $AE = BD$.

Then, since DE is divided into two equal parts in C, and into two unequal parts in A, we have

$DA^2 + AE^2 = 2DC^2 + 2CA^2$ (ii. 9.)

But, since $AE = BD \therefore BD^2 = AE^2$

$\therefore AD^2 + BD^2 = 2AC^2 + 2CD^2$.

(*e.*) The second case of prop. 13 may be proved by means of prop. 7, in the same words, exactly as the first case.

(*f.*) Prop. 11: either here, or at vi. 30; or, still better, at both, show the virtual agreement of both, and that the problem effected, is the cutting a line in extreme and mean ratio (vi. def. 3).

(*Note.* The sections are incommensurable. If the whole were denoted by unity, the greater segment would be ·6180339 +.)

**41.** In book iii. def. 6, as before observed, (art. 36, b.) is the same as def. i. 19.

In this book, let the following hints be attended to: viz.

(*a.*) Let "the angle *in* a segment," and "the angle *of* a segment," (defs. 7, 8), be carefully discriminated.

(*b.*) In the props. whose demonstrations involve the *reductio ad absurdum*, such as props. 1, 2, 4, 5, 6, 10, 11, 12, &c. of this book, and 6, 7, of the first book, the student often meets with difficulty and perplexity in comprehending the *diagram:* let it be explained to him that the difficulty arises from supposing an impossibility possible, or in drawing a figure to suit an impossible case. Let, also, the precise character of this indirect demonstration be amply elucidated.

[Let, also, the student be taught, in all cases, the value of a *correct diagram*, and in all demonstrations involving a construction, let the construction be actually effected.]

(*c.*) Let it be remarked that the 11th def. of this book is lame, insomuch that it rests upon, or takes for granted the 21st prop., which is not yet demonstrated.

(*d.*) Prop. 1. If the fictitious centre, G, be taken in the line CE, the demonstration given does not hold, for in that case no absurdity would follow. Let an adequate correction be supplied.

(*e.*) Prop. 2 may be demonstrated as Commandine has done it, without a *reductio ad absurdum*, by simply assuming as an axiom that, *if a point be taken nearer to the centre than the circumference is,*

*such point is within the circle:* this is, evidently, quite as much an axiom as what Euclid assumes, and it gets quit of the awkward diagram.

(*f.*) Let the student be either informed, or required to show (props. 3 and 4), *why it is a condition that the lines bisected must not pass through the centre:* namely—because if both lines pass through the centre, they *must* bisect each other (def. i. 15), but may be any way inclined to each other.

(*g.*) Prop. 9.—This demonstration is essentially wrong, for, according to it, although D is *not* the centre DB = DA: Hence, any point whatever would appear to be the centre, the point F in prop. 7, for example, which possesses this property.

It may be demonstrated thus: For from any point which is not the centre, only two equal straight lines can be drawn to the circumference (iii. 7); and therefore a point from which *more* than two equal straight lines can be drawn to the circumference *cannot* be any other than the centre. Therefore, &c.

(*h.*) Prop. 11 is incorrectly enunciated; it should be, "if one circle touch another circle internally," they evidently *cannot* touch *each other* internally.

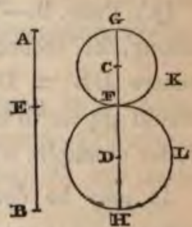
(*i.*) Prop. 14 has the same defect in its demonstration that was noticed in i. 48.

(*k.*) There is an illegitimate assumption in the

demonstration of prop. 20, which should be removed.

It takes for granted a particular case of Euc. v. 5. What has been usually acquiesced in is, that, "if a magnitude be double of another, and if a part taken from the first be double of a part taken from the second, the remainder of the first shall be double of the remainder of the second."

Let AB be double of CD, and let AE, a part of AB, be double of CF, a part of CD, EB, the remainder of AB, shall be double of FD, the remainder of CD.




About the centre C with the radius CF, describe the circle FKG; produce FC to G, FC and CG, that is FG is double of FC (def. 15); but AE is double of FC, therefore FG is equal to AE (ax. 6). Again, about the centre D, with the radius DF, describe the circle FLH, and produce CFD to H; FD, DH, that is FH, is double to FD (def. 15). And, because CF is equal to CG, and DF to DH, CF and DF taken together are equal to CG and DH taken together (ax. 2); that is, CD is equal to CG and DH, and CD, CG, DH, that is GH is double of CD. But AB is double of CD; therefore GH is equal to AB (ax. 6); and GF has been

proved to be equal to AE, therefore FH shall be equal to EB (ax. 3); but FH has been proved to be double of FD, therefore EB shall be double of FD. Therefore "if a magnitude be double of another," &c.

That, "if two magnitudes be double of two others, each of each, the sum and difference of the first two are respectively double of the sum and difference of the other two," is thus shown by *Playfair* in his later editions:—Let A and B, C and D, be four magnitudes, such that $A = 2C$ and $B = 2D$; then $A + B = 2(C + D)$. For, since $A = C + C$, and $B = D + D$, adding equals to equals, $A + B = (C + D) + (C + D) = 2(C + D)$. So again, if A be greater than B, and therefore C greater than D, since $A = C + C$ and $B = D + D$, taking equals from equals, $A - B = (C - D) + (C - D)$, that is, $A - B = 2(C - D)$.

(*l.*) The second case of prop. 21 is more simply proved in Billingsley's Euclid, published in the reign of queen Elizabeth, than in Simson's edition. Thus—

Put I at the intersection of the lines AD and BE; then ABE = ADE (by the first case of the proposition), AIB = EID (by i. 15) ∴ BAD = BED (i. 32).

(*m.*) The demonstration of prop. 26 is inappli-

cable to the case in which the angles A and D are *right angles*.

The difficulty may be obviated, however, by bisecting those angles: their halves are equal, and ∴ the arcs on which they stand are equal.

(*n*.) Prop. 31 admits of a simpler demonstration, thus:

Draw the diameter AEG.

By prop. 20, BEG = 2BAG, GEC = 2GAC

$\therefore$ BEG + GEC = 2 right angles = 2 BAC

$\therefore$ BAC = 1 right angle.

B + A + C = 2 right angles (i. 32)

$\therefore$ ABC < 1 right angle.

∴ angle in a segment > a semicircle is < a right angle.

Draw BD, then angle BAD > BAC > 1 right angle ∴ angle in a segment < than a semicircle is > 1 right angle.

[The converse of the first part of this proposition is assumed in Newton's 11th Lemma, Principia. It may sometimes be given as an exercise for demonstration.]

(*o*.) The addition to this 31st proposition, beginning, "Besides it is manifest," and ending "less than a right angle," only tends to embarrass the student, and had better be omitted.

(*p.*) By means of props. 35 and 36, the geometrical construction of the three ordinary forms of affected quadratic equations, are usually accomplished: as may easily be explained here to an intelligent pupil.

42. In going through the fourth book, the difference between such demonstrations as are subservient to the purposes of pure geometry, and those which facilitate the objects of practical geometry, should be duly explained, as may easily be done by directing the attention of the student to a few of the demonstrations and practical constructions in vol. i. of the Course, pp. 354—371. The references, however, need only be cursory. Occasion may, also, be taken of such references, to show that some of the rules for constructing regular polygons in the ordinary books of practical geometry, are either erroneous, or loosely approximative.

It may, moreover, be well to inform an intelligent, investigating cadet, or other student, that the power of geometry in the construction of regular polygons has been carried beyond that of twelve sides; and that the *septemdecagon*, or 17-sided polygon especially, has been the subject of some elegant and successful researches by *Gauss*, *Lowry*, and others; and that those of Gauss, in particular, were effected before he was 18 years of age.

A very few remarks, in reference to the 4th book, are subjoined.

(*a.*) It ought to be proved in the demonstration of prop. 3, that the tangents to the circle meet at M, N, L; in that of prop. 5, that "lines which are at right angles to parallel lines are also parallel;" in that of prop. 7, that the tangents meet at the points F, G, H, K; and in that of prop. 12, that CK, BK, meet at K, &c.

The method of proof is very easy, and should be supplied. For example, in the figure of prop. 12, join BC; then, because FCK, FBK, are both right angles, the angles KCB, KBC, which are respectively parts of them, are each less than a right angle, and are, therefore, together, less than two right angles: consequently (ax. 12) CK, BK, will meet. The same kind of reasoning applies to LM, KL, and MG, &c.

(*b.*) Let the corollary to prop. 5 of this book be cancelled: it is already demonstrated in iii. 31, and therefore need not have been placed here.

(*c.*) In the demonstration of prop. 15, let the words, "the angles at the base of an isosceles triangle are equal to one another; and," be omitted; they are superfluous.

43. Much that will be proper to remark in connection with the 5th Book has been anticipated in an early part of these observations, when explain-

ing my views as to the best method of teaching the doctrine of *proportion*. (See arts. 18, 19, 20, *a* to *w*.)

To complete the subject, a few more remarks may, however, be given here.

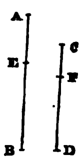
(*a*.) A definition of *equimultiples* should have been given in this book. Let one be supplied in the margin, as below.

"*Equimultiples* of magnitudes are multiples that contain them respectively the same number of times."

(*b*.) Def. 8 had better be modified. What is the meaning of the *similitude* of ratios? If A : B = C : D, there is an equality of ratios; and, since *similitude* can here signify nothing but *equality*, the term is objectionable.

(*c*.) No portion of Euclid's Elements has been so much vitiated and injured by unskilful editors as this 5th book; and it is a portion on which Simson's systematizing labours have been less than usually successful. I shall here simply present two concise and satisfactory demonstrations, in place of the tedious ones retained by Simson.

(*d*.) Prop. 18.




Because AE : EB : : CF : FD.

By alternation, AE : CF : : EB : FD (v. 16).

And, as one of the antecedents : to its

consequent : : all the antecedents : all the consequents (v. 12.) That is, AE + EB : CF + FD : : EB : FD; and by alternation, AB : EB : : CD : FD (v. 16).

(This may be substituted for nearly two pages, in Simson's edition.)

(*e*.) Prop. 20.—The second part of this demonstration is not in the Greek.

This theorem is of no other use than to assist in the demonstration of prop. 22. But that proposition is below done more concisely and easily without it, and, in fact, six demonstrations comprised in one.

(*f*.) Prop. 22 may be demonstrated thus:—Since A : B : : D : E, by alternation, A : D : : B : E (v. 16). Also, since B : C : : E : F, by alternation, B : E : : C : F.

But ratios that are the same to the same ratio, are the same to one another (v. 11). ∴ A : D : : C : F, and by alternation, A : C : : D : F. And, if the number of magnitudes be greater than 3, we may proceed in the same way with the first, third, and fourth magnitudes of each series, as with the first, second, and third; and so on to any number of magnitudes whatever. Therefore, if there be any number of magnitudes, &c., as in the proposition.

(*g*.) Let the distinction between commensurable

and incommensurable quantities be now carefully and thoroughly explained, if not done earlier.

[Quantities are *commensurable* when a third can be found which divides each of them without a remainder, as 6, 24, 9; their common measure being 3.

Quantities are *incommensurable*, when there is no finite quantity that divides or measures each of of them exactly: $\sqrt{6}$, $\sqrt{11}$, or 2, $\sqrt{5}$, $\sqrt{7}$, &c.

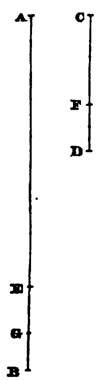
But $\sqrt{6}$, $\sqrt{24}$, $\sqrt{9}$, are *commensurable*, $\sqrt{3}$ being their common measure.

The side and the diagonal of a square are *incommensurable*. (See art. 20, *w*. (5.)

If the geometrical process for finding a common measure of two lines never terminates, the lines are incommensurable.]

(*h*.) To find the numerical ratio of two given straight lines, AB, CD, those lines being supposed to have a common measure.

From the greater AB cut off a part equal to the less CD, as many times as possible; for example, twice, with the remainder BE.




From the line CD cut off a part equal to a remainder of the remainder BE, as many times as possible; once, for example, with the remainder DF.

From the first remainder, BE, cut off a part equal to the second DF, as many

times as possible; once, for example, with the remainder BG.

From the second remainder DF, cut off a part equal to BG the third, as many times as possible.

Continue this process till a remainder occur which is contained exactly a certain number of times in the preceding one.

Then this last remainder will be the common measure of the proposed lines; and, regarding it as unity, we shall easily find the values of the preceding remainders, and at last two of the proposed lines; and hence their ratio in numbers.

Suppose, for instance, we find GB to be contained exactly twice in FD, BG will be the common measure of the two proposed lines. Put BG $= 1$, we shall have FD $= 2$; but EB contains FD once, *plus* GB; therefore we have EB $= 3$: CD contains EB once, *plus* FD; therefore we have CD $= 5$; and lastly, AB contains CD twice, *plus* EB; therefore we have AB $= 13$. Hence, in this case the ratio of lines is that of 13 to 15. If the line CD were taken for unity, the line AB would be $\frac{13}{5}$; if AB were taken for unity, CD would be $\frac{5}{13}$.

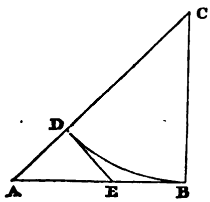
The method just explained is evidently the same as employed in arithmetic to find the common divisor of two numbers (art. 21, *k*. *l*. &c.); and how far soever the operation be continued, it

is possible that no remainder be ever found which shall be contained an exact number of times in the preceding one; when this happens, the two lines have no common measure, and are said to be *incommensurable.*

We may now proceed to show that the *diagonal* and the *side* of A square are incommensurable; and, as the inquiry is instructive, show it two different ways.

Let ABC be the half of a square, the side AB and diagonal AC are incommensurable.

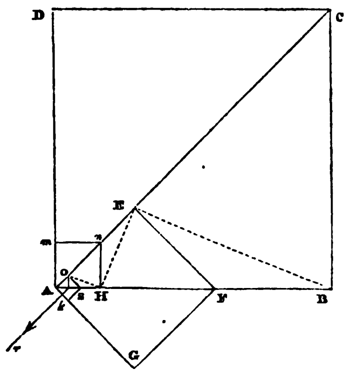
Since the angle at B is a right angle, each of the equal angles at C and A is less than a right angle (I. 32); therefore $AB < AC$. Again, $AB + BC > AC$ (I. 20), or $2AB > AC$; hence AC is greater than once AB, and less than twice AB; and the same may be proved of the diagonal and side of any square. Therefore, when the side of a square is taken once from its diagonal, there is always a remainder less than its side.




From C as a centre, with the radius CB, describe the arc DB; then $CD = CB = AB$, and $AD < AB$. From D draw DE perpendicular to AC; then ED and EB being tangents (III. 16, Cor.), $ED = EB$ (III. 37). But in the triangle ADE,

the angle at D is a right angle, and that at A is half a right angle; therefore that at E is half a right angle, and therefore AD = DE = EB. When the first remainder AD therefore is taken from AB, the remaining part AE, from which AD is still to be taken, is the diagonal of a square, of which AD is the side. But this is the same as the former process; and when it has been performed in regard to AE and AD, the remaining lines to be compared will, therefore, again be the side and diagonal of a still smaller square: but since, when the side of a square is taken from its diagonal, there is a remainder, therefore in the above process there will always be found to be a remainder; the process therefore will never terminate, or no common measure can ever be found; therefore AC and AB are incommensurable.

Mr. *Garrett's* method is this. In which it is shown, that though the successive remainders continually diminish, they still represent the sides of newly generated squares, and retain all the properties of the original which was proposed for investigation.




Let ABCD be a square, AB one of its sides, AC its diagonal: AB and AC are incommensurable.

The side AB is less than the diagonal (i. 19.), and greater than half the diagonal (i. 20,); consequently, the side of a square cannot measure its diagonal.

From the diagonal AC cut off the part CE, equal to the side AB, or BC (i. 3.); join BE, and from the point E, draw EF perpendicular to AC (i. 11.)

The angles CEF, CBF, are equal, each being a right angle; the angles CEB, and CBE, are equal (i. 5); therefore, the angles FEB, and FBE, are equal (i. axiom 3); and, therefore, FE is equal to FB, (i. 6.)

Because, FEA is a right angle, and EAF half a right angle; EFA must be equal to half a right angle (i. 32.), and AE equal to EF (i. 6.), and consequently to BF, which is equal to FE.

Complete the second square AEFG (i. 46.), and make FH equal to AE or BF (i. 3.)

Since AE is less than AF, and greater than half AF, as was shown above, the excess AE, of the diagonal AC, above the side AB of the original square cannot measure the side AB, but may be taken *twice* from the side AB, leaving a remainder AH less than AE.

From the point H, draw H*n* perpendicular to AF, (i. 11.), and complete the third square AH*nm* (i. 46.)

The excess AH of the diagonal AF, of the second square, above its side AE, may be taken *twice* from the side AE, with a remainder A*o*.

From the point o, draw o*s* perpendicular to the diagonal A*n*, (i. 11.) and complete the fourth square A*kso*, (i. 46.)

The excess A*o* of the diagonal A*n*, above the side AH, of the third square, may be taken *twice* from the side AH, leaving a fourth remainder A*r*.

By constructing a fifth square, as above, on A*r*, it can be shown again, that this excess is contained *twice* in the last, with an analogous remainder.

And thus, if the operation be continued, will the same relations be incessantly produced, in such manner, that, although the successive remainders, AE, AH, A*o*, A*r*, &c., constantly decrease, and rapidly converge, yet the process of resolution, or

decomposition, must be interminate. For, since the side of any square cannot measure its diagonal, neither can the excess of the diagonal above the side measure the side; therefore, no common measure of the side and diagonal can be obtained; or, in other words, they are incommensurable.

See another proof in Ivory's Euclid, Incommensurables, props. 3, 4.

The impossibility of finding numbers to express the exact ratio of the diagonal to the side of a square, has now been proved; but an approximation may be made to it as near as we please, by means of the continued fraction, which is equal to that ratio. The fraction is

$$1 + \tfrac{1}{2} + \tfrac{1}{2} + \tfrac{1}{2} + \tfrac{1}{2} + \tfrac{1}{2} + \text{\&c. to infinity.}$$

This fraction is nearly 41 : 29, or a nearer approximation is 10 : 6·9897.

It will be well from the above to impress upon the student's mind that it would be useless to attempt to express accurately by numbers the side and diagonal of a square; and the attempt to give accurate numerical values to the segments of a line divided in extreme and mean proportion would be equally fruitless. We are thus furnished with decisive evidence of the insufficiency of numbers, to

answer rigorously *all* the demands of geometry. We cannot, for instance, take upon ourselves to assert that any two lines that may be chosen at random, shall be susceptible of accurate numerical representation, without first inquiring, whether these lines are commensurable or not; since, for ought we know to the contrary, one of the proposed lines may be equal to the side, and the other to the diagonal of the same square; or the two lines may be the segments of a third line divided in extreme and mean proportion; in either of which cases an accurate numerical representation of the two lines is demonstrably impossible. The diagonal of a square is more than 141421, and less than 141422 parts, when the side is 100000 of the same parts; and, when a whole line of 1000000000 is cut by medial section, the greater segment is more than 618033988, and less than 618033989 of these parts.

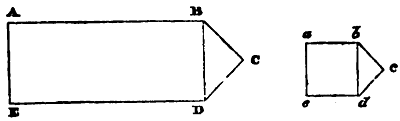
44. The sixth book of Euclid, which treats of the sides and areas of certain rectilinear figures, and contains the investigation of lines that have a proposed ratio to given lines, is of great value to the mathematician.

(*a.*) This book commences with a definition, the elucidatory diagrams to which should be amended. The diagrams are simply triangles, whereas the definition refers to multilaterals, (*i. e.*) to figures of

4, 5, or more sides, as well as to triangles. (See also, pr. 21, 22, 25, 31, in the latter of which, the similar figures in the diagram are rectangles.)

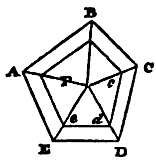
Let the diagrams be modified accordingly.

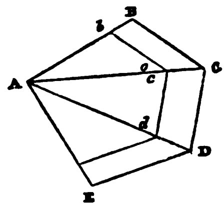
Let the student, also, be required to prove that quadrilateral, or multilateral figures, may be equiangular, and yet *not* similar. In truth, the definition would be better thus: similar rectilinear figures of more than three sides, are those which may be divided into an equal number of similar triangles, similarly situated.




The angles A, B, C, D, E, in the first of these diagrams, may be respectively equal to *a*, *b*, *c*, *d*, *e*, in the second; and yet, the figures are evidently not similar.

Let it be shown, either by the construction of Prob. 47, p. 368, or by some other of the methods, well known to the experienced preceptor, how to effect a precise similarity, agreeably to the definition.




(*b.*) In the demonstration of prop. 1, which is an elegant application of the principle of def. 5, book v. After the words "point A to BD," add, "or BD produced."

(*c.*) To complete the logical series of propositions in this book, it should have been shown that triangles and parallelograms, having equal bases, are to each other as their altitudes. The omission may be correctly supplied by means of Theor. 80, p. 323, vol. i. of the Course.

(*d.*) Propositions 14, 15. The present demonstration of prop. 15, which is often perplexing to learners, may altogether be superseded by a slight modification of the demonstration of prop. 14.

Join DF, FE, EG, by right lines; then substituting the triangles DBF, FBE, EBC, for the parallelograms AB, FE, BC, in the demonstration of prop. 14, it serves throughout for the demonstration of prop. 15.

(*e.*) Proposition 18. What is *similarly situated?*

And is *position* an essential condition in the construction?

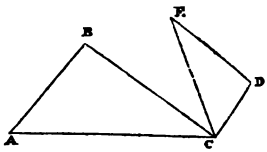
(*f*.) Propositions 28, 29, need not perhaps be dwelt upon, *now;* but the student should be informed that they were absolutely necessary to the ancients, that they might effect geometrical operations, analogous to the algebraic solution of quadratics:—

Prop. 28, of such as $x^2 - ax + ml = o$:

Prop. 29, $x^2 - ax - ml = o$.

(*g*.) Prop. 25, is the same as the problem annexed to prop. 60 of the data. Let the student be required to compare them, and decide which of the demonstrations is to be preferred. (See also, Simson's note to prop. 25, of this sixth book.)

(*h*.) The enunciation of prop. 32, and of part of the demonstration, is vague:—the first defect may be removed, by adding to "one another" in the proposition, the words "and so that the sides that are *not* homologous, form the angle at which they are placed;"—and the second, by annexing to the words "let AB be parallel to DC, and AC to DE," the following, "and AC, CD, the sides not homologous from the angle ACD."



[Here the homologous sides AB, DC, are parallel; but the other sides do not form a right line.]

(*i.*) Let prop. 33 be studied with the prospective reference to the measure of areas and angles, already alluded to, art. 9: and let the value of propositions B, C, D, in trigonometrical researches be carefully pointed out.

45. On passing from the plane geometry, to that of solids and of intersecting planes, let not only the reason for the transition from Euclid to Hutton be assigned (see art. 35); but the essential difference between the two methods, when applied to the capacities of solid bodies, be amply stated and elucidated.

Euclid's method is that of *exhaustions*, of which a beautiful example at the very commencement of Book xii., should be explained.

The method adopted by Hutton, is that of *indivisibles*, invented by Cavalerius: its most elegant application is to Theor. 122; of which, however, the logical defects should be pointed out.

46. In the demonstration of Theor. 114, the student should be required to examine, with a view

to its application to the case in hand, the principle of the notes at pages 319, 322 (or Theor. 67, 79); and he must not be permitted to quit that proposition, until the master is satisfied that he has seized and comprehended the principle unfolded in the notes cited, so as to be able to apply them to all analogous cases.

47. In order that the comprehension of this part of geometry may be facilitated, the master should avail himself of models of intersecting planes, and of different solids; constructed of wire, pasteboard, wood, or metal, as may best suit the several cases.

48. Before the student passes from the study of the geometrical portions of the Course, and the Elements of Euclid, let him, that he may complete his summary of the latter, be informed of what the remaining books mainly consist.

(*a.*) The 7th book contains 41 propositions. The 8th, 27. The 9th, 36.

In the 7th book, proportion is defined as it regards commensurable quantities; and, in that restricted sense, applied with some adventitious aid, in the 7th, 8th, and 9th books, to investigate the properties of odd, even, prime, composite, plane, solid, square, cubic, perfect, and imperfect numbers.

The 10th book contains 117 propositions, in

which are announced and contrasted, the properties of commensurable, and incommensurable, rational and irrational quantities; and further investigated several properties of their roots and powers.

The 11th and 12th books are in Simson's editions; and need not be here described.

The 13th book comprises 18 propositions. The 14th, eight. The 15th, five. They relate to the higher, and in some respects connected and dependent properties of circles, polygons, and the five regular bodies.

Flussas and others, as Sir Henry Billingsley in his edition before-mentioned (art. 41, *k.*), present supplementary disquisitions, on many subjects which were of great interest when the whole of mathematical knowledge related principally to geometry, but which have naturally diminished in importance and interest, as this department of intellectual inquiry has more widely diffused itself.

49. The subject of *Trigonometry* is usually divided into two branches; *ordinary* and *analytical.* The first comprehending the solution of *triangles;* the second, while it also comprehends that, is extended greatly farther, to the investigation of the numberless properties of arcs, angles, and linear angular quantities, whether in their separate or complex relations.

[This extended application of the term, furnishes another example of the manner in which enlargements of the field of knowledge occasion modifications in the use of terms.

Modern Algebra presents one of these, in the extension given to the meaning of the word *addition.* (See note, p. 116, vol. i. Edition 11.)

Geometry presents another, even in its very designation: for the term *geometry*, when first used, simply implied the measure of *the earth;* or rather, of the portions of *land* in Egypt, which were rendered doubtful by the periodical inundations of the Nile, which often swept away the land-marks.

By a like modification, the word *trigonometry*, which first denoted the measure of triangles, and their several parts, now includes the investigation of magnitude in a state of alternating increase and decrease, or of periodic magnitude, (art. xx. 5.)

50. The trigonometry, in vol. i. of the Course, is confined principally to the solution of triangles, and its practical application to the important subject of the determination of heights and distances.

It should be taught—

(*a.*) So as to ensure to the student a thorough comprehension of the several demonstrations and investigations.

(*b.*) So that he may acquire facility and correct-

ness in the logarithmic and other operations. With respect to which, the tables of verification, both with regard to plane triangles and spherical triangles, at the end, may enable the instructor to vary the cases to the utmost variety, and with the answers before him.

(*c.*) So that he may exemplify the relative advantages and disadvantages of log. tables, carried to 5, 6, 7, or more, places of decimals, and to what extent, and on what occasions, even 5 places may be safely employed.

(*d.*) So as to show that in some cases, and in what, the arithmetical operations, by natural sines, tangents, and secants, may arrive at the result required, quicker than by the logarithms.

(*e.*) With such attention to accuracy of construction, that the diagram, drawn previously to the computation, may point out when the result is ambiguous, and mark the character of the ambiguity; as in the case of example 3, and 4, p. 394.

(*f.*) So as to bear in memory for prompt recollection, (art. 6.) the most useful trigonometrical formulæ, (p. 401—405).

[In the 29th, it will be well to deduce the same simple proportions for the logarithmic operations, both from a geometrical and an analytical process;

that their respective powers in such researches may be compared.

The student may be also referred to some of the analytical solutions, of problems in heights and distances, in the 2nd volume.]

He will again often find it very useful, in extending his knowledge of trigonometrical transformations, to become familiar with Buck's table of trigonometrical multiplication and division, inserted in the supplementary tables of this volume; and so, as opportunity serves, to go carefully over the analytical trigonometry in the 2nd volume, and the application of trigonometry to the solutions of equations, in the appendix to Mr. Hind's neat work, on the "Elements of Trigonometry." Some simple problems, for preliminary trials, may be easily chosen by the preceptor.

For example:

In a plane triangle, given that

$$b = a \sin. C, \; c = a \cos. B.$$

Find its angles.

Multiply the second given equation by $2c$,

$$\text{then } 2c^2 = 2ac \cos. B;$$

but in all triangles $b^2 = a^2 + e^2 - 2ac \cos. B$,

hence by addition $b^2 + c^2 = a^2$;

and, therefore, the triangle, is right-angled, $a$ being the hypothenuse; then $A = 90°$,

$$\text{and } B + C = 90°;$$

consequently, sin. C = cos. B,

or $\frac{b}{a} = \frac{c}{a}$, and $b = c$;

or the triangle is isosceles, and B = C = 45°.

To gain expertness in such transformations, another pupil may be required to furnish a different solution: For example,

$$c = a \cos. B;$$

therefore, $o = b$ cos. A, or cos. A $= o$, and A $= \frac{1}{2}\pi$,

Similarly, $b = a \cos. C + c \cos. A$

$= a \cos. C$, since A $= \frac{1}{2}\pi$,

$= a \sin. C$, by the question;

then sin. $c$ = cos. C, and C $= \frac{1}{4}\pi$.

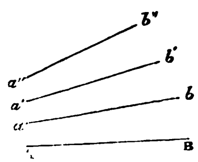
Lastly, B $= \pi - (A + C) = \pi - \frac{3}{4}\pi = \frac{1}{4}\pi$, and the triangle is right-angled and isosceles.

51. Although analytical trigonometry, and its principal applications, will usually be referred to the instructions of one of the higher mathematical masters; yet, it will not be amiss, with the *prospective* reference which I have more than once characterised as highly valuable in intellectual instruction, for the masters in the lower rooms, to give to an intelligent pupil, in his course through the ordinary trigonometry, some such illustrations of the nature of periodic magnitude, as linear angular quantities, or the trigonometrical functions, sin., cos., tan., cot., &c. most naturally supply.

"Taking the primary idea (says Mr. De Morgan,) of quantity, alternately increasing and de-

creasing, it is obviously of fundamental importance, to detect a proper method of measurement. The circle presents itself for the purpose, in the following way. Conceiving a periodic change of magnitude to run through its whole cycle in a given time, let a point revolve uniformly round a circle in the same time, starting from the end of a fixed diameter. The height of the point above the diameter is a *periodic* magnitude, which goes through all its changes in the same time as the given magnitude; and it is, in fact, one of the great objects of the higher trigonometry, to express periodic variation, whose law is known in any way by means of this simple species of variation."

It is evidently from geometrical magnitudes alone that we can select those which are necessarily periodic, and altogether exclusive of indefinite increase: the specific idea is conveyed in the term *direction*; and the consideration of varying direction (as in $ab$, $a'b'$, $a''b''$, &c. referred to the fixed position AB), and consequently of periodic magnitude, constitutes the connecting links of geometry with physics, a




connection which has been indissolubly established principally by the successive and rapid improvements in analytical trigonometry.

52. It may serve as a powerful stimulus to an intelligent cadet, to be informed that the most important step in the trigonometrical analysis was made 80 years ago, by T. Simpson, then professor of mathematics in this institution; that step, I mean, which, instead of representing the sine, co-sine, tangent, co-tangent, &c., of any and every arc or angle by *a separate letter*, avoided that source of perplexity and embarrassment altogether, by using the abbreviations, sin. A, cos. A, tan. C, cos. 2 B, cos. 4 B, cos. (P — Q), &c.[1]; and farther, that, in consequence of this beautiful facilitating process, he made *the first step*, after Newton, in the series of discoveries in physical astronomy, which has been followed up so admirably by *Bernoulli*, *Euler*, *Frisi*, *Lagrange*, and *Laplace*.

Simpson, in his researches into that part of the celestial physics which relates to the moon, having shown that no terms enter into the equation of the orbit but what are expressible by the cosine of an arc or the cosines of its multiples, and, therefore, that no terms enter that equation but what, by a regular increase and decrease, return to their former

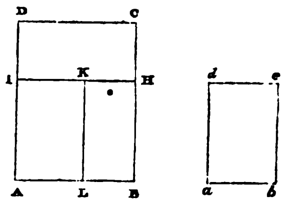
[1] This elegant and valuable improvement is often ascribed to *Euler*, (see Dean Peacock's "Report on Certain Branches of Analysis," "Report to the Third Meeting of the British Association," p. 289); but it is certainly due to Simpson, and claims a priority of many years.

values; immediately infers irrefragably, that all the variations in the motions of that body are only so many oscillations between certain limits, and therefore, "that the moon's mean motion, and the greatest quantities of the several equations, undergo no change from gravity." (Miscellaneous Tracts, published in 1757.) This may occasionally be enforced as a remarkable proof of the advantages which may flow from a simplification of notation.

53. On passing to the subject of mensuration, and its practical applications (pp. 414—432, 453—468), the first object should be to give a firm root to the persuasion that the connection between products in numbers and the areas of rectangles in geometry does not repose upon a bare and loose analogy, but upon scientific principle. Perhaps, therefore, it will be advisable, instead of allowing the student to rest satisfied with the note at p. 415, or the corollaries to Theor. 81, Geom. (p. 324), there referred to, to require him to take a closer view of the matter. Let him, for example, study the subjoined four propositions (*a*), (*b*), (*c*), (*d*).

Rectangles are to each other as the *products* of the numerical measures of their bases and altitudes.

Let AC, *ac*, be two rectangles, with bases AB, whose measure is B, and *ab*, whose measure is *b*; and heights AD, measure H, and *ad* measure *h*.




Suppose one of these figures applied to the other, so that *ab* shall coincide in direction with AB, and with AI or part of AD, and the right angle *a* with A. Thus we shall have the rectangles AK denoted by $r$, and AH, by R′, of the same altitude AI; and the latter having the same base as the rectangle AC = R: we therefore have, by known principles, not here specifically quoted,

$$\begin{array}{rl} & R : R' :: H : h \\ & R' : r :: B : b \\ \therefore & R : r :: BH : bh \\ \text{or} & \dfrac{R}{r} = \dfrac{BH}{bh} \end{array}$$

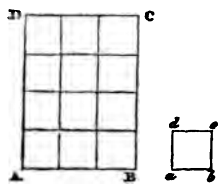
The method would not vary essentially, if *b* were greater than B.

The same theorems are equally true for parallelograms generally, since they are respectively equivalent to the rectangles of the same bases and altitudes.

Hence, parallelograms are to each other as the products of their bases and altitudes.

(*b*.) Farther, to measure an area is to determine

the number of times that it contains some other given area. Thus, to measure the rectangle ABCD, let *abcd* be taken for the unit of surface. Since, then, by the above, $\frac{R}{r} = \frac{B}{b} \times \frac{H}{h}$, we shall suppose the measure of *ab* carried along AB, in order to ascertain how often the one is contained in the other;



then the same done with regard to the altitudes *ad*, AD; and then the product of the numbers taken. In the case of the figures, this would be $3 \times 4 = 12$, and in this case R would be $r$ twelve times.

But, since the bases and altitudes may not always contain each other precisely, it is still better to regard the measure of an area ABCD as its ratio to another, *abcd*, taken for unity; and this measure is evidently the product of the ratio $\frac{B}{b}$ of the bases, by the ratio $\frac{H}{h}$ of the heights. The same obtaining for every parallelogram, it follows, that if $n$ represent the abstract number $\frac{B}{b} \times \frac{H}{h}$, the area of the parallelogram is $n$ times that which is taken for the unit of surface.

(*c.*) If for this unit of surface we take the square

*abcd*, whose side is the linear unit, then we have $b = h = 1$; whence $R = BH$. BH is the product in abstract numbers of the respective numbers of lineal units contained in B and H; and if this product be assumed also equal to $n$, we have $R = n$ times the square assumed for the unit of area. Hence, *the area of any parallelogram is the product of the numbers of times that the linear unit is contained in its base and in its altitude respectively:* this is otherwise often expressed in the concise though inaccurate terms, *the area of any parallelogram is the product of its base and altitude.*

(*d.*) When the sides of the rectangle are equal, its measure is $BC \times BC$; so that the area of its square is the *second* power of its side. It is hence that the terms *square* and *second power* are regarded as synonymous.

*Cor.* If the side of a square be $s$, where $s$ is either whole or fractional, the square itself is $s^2$, and *vice versâ.*

Hence the existence of incommensurables may be proved. For, if the side of an equilateral rectangle be commensurable with the linear unity, let it be $s$; then $s^2$, the number denoting how many square unities are contained in the rectangle, admits of a square root; therefore, conversely, in case this number does not admit of a square root,

the side is not commensurable with the linear unity. (See *Francœur's Course.*)

54. The general subject must be pursued, consistently with the principles it has been the entire purpose of this paper to elucidate and enforce.

The best practical methods of operation must be explained and exemplified: checks and verifications pointed out in cases where they are known to exist: the notes for demonstration must be thoroughly studied, and the instances specified (though, of course, only by anticipation here), in which the direct and inverse method of fluxions, or the investigations of the differential and the integral calculus yield their analogous results by a more elegant process.

The inventive and investigating faculty of the student may here, again, be kept alive, by requiring occasionally an independent investigation, or a deduction not specified in the book. Thus he may be required to show—

(*a.*) That, when a triangle is equilateral, or $a = b = c$, the rule in note p. 417, then becomes area $= \frac{1}{4} a^2 \sqrt{3}$.

(*b.*) When it is isosceles, or $c = b$, then area $= \frac{1}{4} a \sqrt{(2b + a)(2b - a)}$.

(*c.*) That, when of the four sides of a trapezoid,

*a*, *b*, *c*, *d*, *a* and *c* are parallel to each other, and $c - a = \delta$, then,

$$\text{area} = \frac{a+c}{4\delta}\sqrt{(b+d+\delta)(b+d-\delta)(b+\delta-d)(d+\delta-b)}$$

(*d.*) That, when the two diagonals *a* and *b* of a quadrilateral intersect at an angle $\theta$, then,

$$\text{area} = \tfrac{1}{2}\, ab \sin.\ \theta.$$

(*e.*) Not a few of the miscellaneous exercises at the end of the volume will be found of value both with respect to practice and to theoretical deduction; a judicious selection of them must, therefore, be made, and perhaps a few new ones be added[1].

[1] Thus, suppose it became necessary to ascertain the radius of a very slender cylinder, you might teach to proceed thus:—

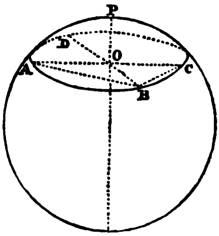
Fix to the extremity of a very fine and flexible line a weight sufficient to keep the line stretched; fasten the other extremity to the axle of which the radius is required: the line being stretched by the weight mentioned above, measure by a scale of even parts, any convenient length, (six inches, for example), and mark the extremities of the length so measured; then, holding the axle horizontal, let the measured part of the line be wound round it in the form of an helix, the circumferences being every where contiguous. Count the number of complete revolutions, and suppose them $= n$; also measure the length of the cylinder occupied by the helix; let this $= a$, and the length of the helix or line first mentioned $= l$, $p = 3{\cdot}14159$, &c., then, the radius of the cylinder

$$= \frac{\sqrt{l^2 - a^2}}{2pn}.$$

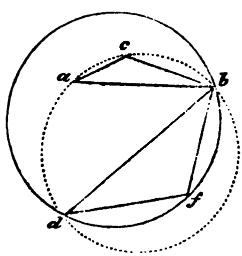
The exactness of this method may be known by observing, that if the cylinder be truly made, and the process carefully repeated, with different values of *l*, *n*, and *a*, the radius deduced will, how-

Indeed, in going over the business of Mensuration, an intelligent student will not satisfy himself by merely being put in possession of the working

ever, always come out the same to the 4th or even the 5th decimal place. This method is due to Mr. Atwood.




Again, suppose it were required to find the diameter of a sphere or ball of ivory, or of iron, or of copper, from any point whatever P, of the sphere, describe about the sphere a circle ABC; imagine the three points A,B,C, or measure with the compasses the distances AB, BC, AC, and draw a triangle *abc* of precisely the same size; if on this triangle *abc*, we describe the circle *abcd*, it will be precisely the size of the circle, that is, the smaller circle of the sphere. If we conceive the point P as the centre of the circle ABC, it is plain that PO is perpendicular to the plane of the circle, and that this right line PO will pass through the centre of the sphere; therefore, the circle passing the three points B, P, D, will be a great circle of the sphere. Therefore, we shall have as O P : the right line or distance PC :: PC : the diameter of the sphere. Or, we may proceed thus: draw a diameter *bd*; from the point *b* as centre, with a radius equal to PB, describe an arc of a circle, and from the point *d* as centre with the same radius describe another arc which will cut the former in *f*. The triangle *bdf* will be equal to the triangle BPD; therefore the circle *bdf* will be a great circle of the sphere given; we shall know, therefore, the diameter of the sphere required, by either of these methods.



rules, but will find it alike pleasing and profitable to be put into the capacity of selecting and retaining several theoretical and practical deductions for use in after life; such, for example, as are inserted in the preceding note will be beneficial.

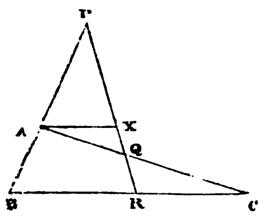
We annex, also, a few propositions, taken from Mr. *Brianchon's* theory and application of transversals, which will be of great utility to officers of artillery and to civil and military engineers.

1. A piece of artillery being placed in A, and striking the point P, required to ascertain the distance AP; in this case the range.

Having taken on the earth, at pleasure, a point which is in a right line with A and P, and a point R out of that line, prolong BR by any quantity whatever RC; then marking the point Q, where the right line AC is cut by PR, we have, by the theory of transversals,

$$PA \,.\, QC \,.\, RB = PB \,.\, QA \; RC.$$

Whence we find $PA = \frac{PB \,.\, QA \,.\, RC}{QC \,.\, RB} \therefore \frac{PA}{PB} = \frac{QA \,.\, RC}{QC \,.\, RB}; \frac{PA}{PA - PB} = \frac{QA \,.\, RC}{QA \,.\, RC - QC \,.\, RB},$

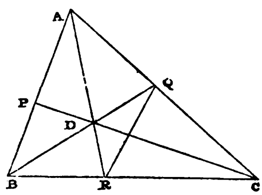



and $PA = AB \,.\, \frac{QA \,.\, RC.}{QA \,.\, RC \backsim QC \,.\, RB}.$

We can thus determine the breadth of a river, over which suppose it is required to throw a bridge: let A be a signal placed on the accessible bank; P a remarkable point on the opposite bank.

II. It is required, through a given point, to draw a parallel to an accessible line.

Let Q be the given point, and AB the accessible line. Having drawn at pleasure, from the point Q, a right line cutting the given line in A, take on AB any length AP, carried from P to B; then from a point C, taken arbitrarily on the line AQ, draw the right lines CP, CB, and mark the point D where BQ cuts CP; finally, draw AD,

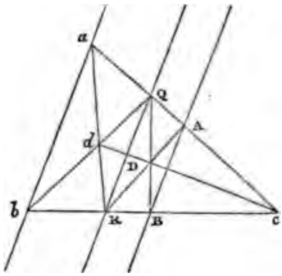


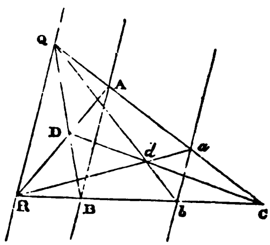

which meets CB at R, and QR is the required parallel.

This solution depends upon this, that the base of a triangle being divided in two equal parts, by a right line drawn from the summit; if from any point of this right line we let fall, on each side, a transversal passing by the opposite angle, the feet of these two transversals will determine a line parallel to the base.

III. Through a given point to draw a parallel to two lines parallel respectively.

Here QR, *ab*, are the two parallels, and A the given point. From the point A having drawn at pleasure a right line which intersects the first parallel in Q, and the second in *a*, we take on this right line any point whatever C, where we draw a new arbitrary right line, which cuts the parallels in R, *b*, respectively; mark the point *d* where the two lines Q*b*, R*a*, intersect; then the point D of intersection of C*d* RA; lastly, we draw QD which intersects CR in B, and AB is the parallel required.




No chain is here required. If Q, R, *b*, are remarkable points, we may suppose them inaccessible. Whether the point given is placed between the parallel or not, the method will be the same.

IV. To divide a line in two equal parts, AB being the given line, we draw a parallel QR (the

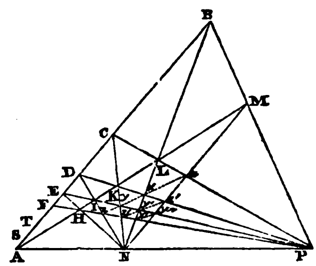
figure being that of prob. II.), and we draw towards any point whatever C, the lines AC, BC, which intersect that parallel in Q and R; then we place at the point D of intersection of the two lines BQ and AR, the line CD prolonged will cut AB in P in two equal parts.

The points A and B may be inaccessible, it will then be necessary to place a short pole or picket between the two inaccessible points, and in their "alignement," which is a very simple practice.

V. To divide a given or known right line into any number whatever of equal parts.

AB is the given line. Make, at pleasure, a quadrilateral, one of whose diagonals is parallel to AB, and whose opposite sides meet in A and B, respectively; the other diagonal cuts AB in the middle C, making,

$$AC = \frac{AB}{2}.$$




Make, at pleasure, a second quadrilateral, one of whose diagonals is drawn to B, and whose opposite sides meet in A and C, respectively; the other diagonal cuts AC in a point D, and we have,

$$CB : CD :: AB : AD, \text{ or}$$
$$AB - AC : AC - AD :: AB : AD$$

making, $$AD = \frac{AB}{3}$$

Make, at pleasure, a third quadrilateral, one of whose diagonals is drawn to C, and whose opposite sides meet in A and D, respectively; the other diagonal cuts AD at a point E, and we have

$$DC : DE :: AC : AE$$

or, $$AC - AD : AD - AE :: AC : AE$$

making, $$AE = \frac{AB}{4}.$$

Make, at pleasure, a fourth quadrilateral, one of whose diagonals is drawn to D, and whose opposite sides meet in A and E respectively; the other diagonal cuts AE at a point F, and we have,

$$ED : EF :: AD : AF,$$

or, $$AD - AE : AE - AF :: AE : AF,$$

making, $$AF = \frac{AB}{5}.$$

We may thus continue to deduce each segment, $\frac{AB}{6}$, $\frac{AB}{7}$, from the two preceding segments.

The system of quadrilaterals of construction may be simplified thus:

AB being the line which is to be divided in equal parts; draw at pleasure AP; and, through any point N on AP, draw NM parallel to AD, in-

tersecting BP at M; let the point L, of intersection of AM and BN be marked; and draw PL intersecting AB at C, and we have,

$$AC = \frac{AB}{2}.$$

Let the point K, of intersection of AM and CN be marked; draw PK, intersecting AB at D, and we have,

$$AD = \frac{AB}{3}.$$

Mark the point I, of intersection of AM and DN; draw PI, intersecting AB at E, and we have,

$$AE = \frac{AB}{4}.$$

Mark the point H, of intersection of AM and EN; draw PH, intersecting AB at F, and we have,

$$AF = \frac{AB}{5}.$$

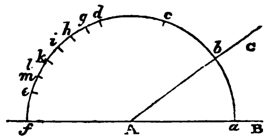
From this it will be easy to proceed to obtain any segment, as $\frac{AB}{6}, \frac{AB}{7}, \ldots \ldots \frac{AB}{m}$.

Before the subject of mensuration, and its more useful applications to practical men, is entirely quitted, it will be well to draw the attention of the student to Mr. De Lagny's curious process for successive approximations to circular arcs, first announced to the public about 1724, under the name of *Goniometry*. It will be easily comprehended

by giving a single example. The method, in fact, consists in measuring an arc or angle proposed with a pair of compasses, without any scale whatever except an undivided semicircle. Produce one of the sides of the angle backwards, and then with a pair of accurate compasses describe a sufficiently large semicircle, from the angular point as a centre, cutting the sides of the proposed angle, and thus intercepting a part of the semicircle. This intercepted part is accurately taken between the points of the compasses, and turned successively over upon the arc of the semicircle, to ascertain how often it is contained in it, and the remainder, if, as usual, there be one, marked; then take this remainder in the compasses, and in like manner find how often it is contained in the last of the integral parts of the first arc, with again some remainder; find in like manner how often this last remainder is contained in the former; and so on continually, till the remainder become too small to be taken and applied as a measure. By this means Mr. De Lagny obtains a series of quotients, or fractional parts, one of another, which being properly reduced into one fraction, give the ratio of the first arc to the semicircle, or of the proposed angle to two right angles, or 180 degrees, and consequently that angle itself nearly in degrees and minutes.

Thus, suppose the angle BAC be proposed to be

measured. Produce BA out towards $f$; and from the centre A describe the semicircle $abcf$, in which $ab$ is the




measure of the proposed angle. Take $ab$ in the compasses, and apply it 4 times on the semicircle as at $b$, $c$, $d$, and $e$; then take the remainder $fe$, and apply it back upon $ed$, which is but once, viz. at $g$; again take the remainder $gd$, and apply it 5 times on $ge$, as at $h$, $i$, $k$, $l$, and $m$; lastly, take the remainder $me$, and it is contained just 2 times in $ml$. Hence the series of quotients is 4, 1, 5, 2; consequently the 4th or last arc $em$ is $\frac{1}{2}$ the third $ml$ or $gd$, and therefore the third arc $gd$ is $\frac{1}{5\frac{1}{2}}$ or $\frac{2}{11}$ of the second arc $ef$; therefore again this second arc $ef$ is $\frac{1}{1\frac{2}{11}}$ or $\frac{11}{13}$ of the first arc $ab$; and consequently this first arc $ab$ is $\frac{1}{4\frac{11}{13}}$ or $\frac{13}{63}$ of the whole semicircle $af$. But $\frac{13}{63}$ of $180^{\circ}$ are $37\frac{1}{7}$ degrees, or $37^{\circ}\ 8'\ 34''\frac{2}{7}$, which therefore is the measure of the angle sought. When the operation is nicely performed, this angle may be within two or three minutes of the truth.

It may be added, that the series of fractions

forms, in fact, a continued fraction. Thus, in the example above, the continued fraction, and its reduction, will be as follows:—

$$\cfrac{1}{4+\cfrac{1}{1+\cfrac{1}{5\frac{1}{2}}}}=\cfrac{1}{4+\cfrac{1}{1\frac{2}{11}}}=\frac{1}{4\frac{11}{13}}=\frac{13}{63};$$

the quotients being the successive denominators, and 1 always for each numerator.

A similar method is applied in *Geo. Adam's* book on mathematical instruments, with great ingenuity and success, to successive approximations of the portions of a right line.

Mr. Sankey's recent invention of the *Cyclometer*, consisting of a metallic cycloid, with its base and axis, and the diameter of the generating circle, will very advantageously and accurately apply to the measure of circular arcs.

55. It will, doubtless, have been observed that, with respect to prospective allusion, I have but seldom touched upon matters connected with mechanics or physics. With regard to this, therefore, I would remark, once for all, in the language of Mr. Whewell, that "the student cannot advance in this field steadily or clearly, except he *feel distinctly that he is on new ground;* that he is dealing with a set of notions different from those of pure

mathematics, and resting on peculiar principles. Of this, perhaps, he is not always apprized. He is led, or left to imagine, that his new study is merely a developement of his notions acquired in the previous parts of mathematics." "If he approach the subject with such an impression, it will be no wonder if his notions always remain mere algebraical abstractions, without mechanical value or meaning; and if he himself continue to the end of his career, incapable of applying them to any really mechanical question."

These views are in perfect accordance with what I have long held, and often maintained in my own lectures on Natural Philosophy. It is gratifying to find them supported by so high an authority.

56. I may now bring these remarks to a close. My object has been to show to what purposes the mathematical studies in the two lower rooms of our Institution must be directed, and with what spirit they must be conducted by the masters and pursued by the pupils, and similarly among the lower classes in most other seminaries, so that the result may be permanently beneficial. Assigning to education its specific object, and to mathematics its due place, in a system of liberal education, I have attempted to mark the distinction between the retaining and the recalling functions of me-

mory, and to show how both may be improved and extended by a judicious use of the law of association. To this end I have remarked that scientific truths must be deposited in the mind, as far as possible by a classified grouping, upon principle, and with a prospective reference; that they must be received, not inertly, but with an attention stimulated and kept alive by a sense of their value; that passive habits must be replaced by *continuous* mental activity and persevering search after knowledge; that the successive topics of instruction must be communicated and received, not as insulated independent fragments, but as parts of a whole; that each and all must be thoroughly comprehended, and permanently secured by frequent retrospection and revision; while the faculty of *looking* forward, and the intellectual instruments for *going* forward, are gaining fresh strength from every new acquisition. I have, then, sketched in detail, but with as much brevity as was consistent with perspicuity, my own view of the manner in which the rudimental portions of the Course may be systematically blended and successfully taught. I hope I may anticipate the cheerful concurrence of both masters and cadets in the complete working out of a plan which the extended pre-requisites of admission now render practicable;

and thus prepare for mature life, not mere routine passive agents, but reasoners and active men; sound, clear, conclusive reasoners; judicious, persevering, well-instructed, successful men.

OLINTHUS GREGORY,

*Professor of Mathematics.*

22nd July, 1837.

# POSTSCRIPT.

Since the completion of the preceding paper, I have seen the "Report of the Regents of the University of New York," March 1837, in which are skilfully embodied, in small compass, the leading principles of sound instruction, in three comprehensive practical observations, which, on account of their obvious application to the purposes of such an Institution as ours, I subjoin.

"The leading principles to be thoroughly inculcated, are—

"1. *Exactness:* by which is understood, the learning perfectly whatever is professed to be learnt at all.

"This can never be attained without attention and patience—causing the subject to pass and repass in close and frequent examination, till it become familiar, and leave an indelible impression

on the mind. The exciting of such a habit of attention, as it is the first duty, so it is the greatest difficulty, and the most important victory of an able teacher, and the cardinal secret of sound education. To produce it, he must insist, peremptorily and inexorably, upon *exactness*. His pupils will shrink, they will solicit, they will complain; they may feel a momentary despondence; but there is an elasticity in youth, which cannot be long depressed, and a generosity which the firmness of authority, softened by a well-adapted soothing, can work up to astonishing efforts." "The contrary course terminates in the worst effects. Let a youth 'get along,' as the phrase is, 'pretty well;'—let his ideas on a point, or his acquaintance with a subject which he is required to master, be only general and confused—let him conjecture where he should be certain; let his preceptor almost put the answer into his mouth, when he hardly knows which way to guess, and he is bribed to intellectual sloth; the season in which he should fix habits of discrimination, as well as of prompt acquisition, passes by; and though he bring to the college good native powers, he will leave it with a mind inert and unproductive. Let the idea, then, of a medium between scholarship and no-scholarship, be for ever banished. Let the ideas of doing a thing, and doing

it well, be identified in the minds of both preceptor and student; and let the doing a thing by halves, be equivalent to not doing it at all.

"2. *Punctuality:* by which is meant, that the performance of all exercises should be limited to a certain time, and then be rigorously exacted. The teacher will, of course, take care that they be both reasonable and sufficient. Under these conditions, nothing but a physical impossibility, or such a hindrance as cannot at all be referred to indolence or evasion, should excuse for non-performance. Exactness is not to be expected without close labour. Labour will not be regular and ardent, without the pressure of necessity. Let it soon be ascertained that there is no escape—that the thing *must* be done—and it *will* be done. Such an urgency upon the mind disarms temptations to trifling, and often to vice; keeps it bent on the period and the matter of duty, and throws it into strong, and at length spontaneous action; hence spring the finest effusions of human genius. There exists no more fatal enemy to diligence, improvement and excellence, than the persuasion that 'there is time enough.'

"3. *Progression:* by which is understood a graduation of exercises, from easier and shorter, to more difficult and ample, according to the power of performance.

"During the whole course of education, the faculties are to be kept on the alert. As they develope themselves, they are to be employed in work demanding higher tension, and more dauntless vigour. As in mathematical science, every preceding proposition, is an instrument in the demonstration of those which follow; so, in all the branches of education, every thing which, before being learned, is an *end*, becomes, when learned, a *means*, and is to be applied in its turn to the remoter, and abstruser investigations. On no account, therefore, ought students in the more advanced classes to spend their time merely in those elementary studies which occupy beginners. It is the impoverishment of intellect,—it is a waste of life,—it never *can* be necessary, unless the necessity be created by some mismanagement in the system."

Further, as well with regard to the distribution of time, with a view to the acquisition of mental strength, as to that of mental riches; avoid usually the extremes on the one hand, of learning only *one thing;* and, on the other, of learning *every thing*. Forget not the excellent precept of the great *De Witt*, so strongly eulogised by *Dr. Waring*. *Do only one thing at once, attend only to one thing at a time.* He who is learning, should learn one thing at a time, and not suffer the thoughts to be

diverted too rapidly, from one study to another, without order, method, or sequence; he may thus lose every thing. And so, with regard to the teacher, it is ordinarily best to teach one thing, and one person at a time, and often to teach in a class, that he may well appreciate the advantages of that order of tuition.

And, in reference to the acquisition of knowledge generally, let it be recollected that it is seldom, if ever, right to *confine* it to one subject, or one department of knowledge. If, for example, a student is aiming to acquire the knowledge of caloric and the gases, he must not permit physics to be separated from chemistry; if he aims at a deep insight into geology, he will not so exclusively devote himself to mineralogy, or conchology, as to abandon astronomy, and especially the celestial dynamics; or, to exclude the wonderful nebular hypothesis, of which so much of special interest may be collected from the works of *Laplace*, *Whewell*, and the *Herschells*. All the natural sciences form almost infinite ramifications, and seem in a manner to constitute portions of one and the same science; so that in the pursuit of knowledge, it will frequently happen that several sciences nearly meet at some great truth, and there is scarcely any which is not connected with others by points of contact, more or less numerous: hence, bearing

this in mind, students may preserve themselves from the vanity and pedantry of half-knowledge, by apprehending *with accuracy*, what they actually learn: and, in order to ensure their ultimate conquests, making in succession their positions in whatever regions they enter, with firmness and safety.

And, with a view to all the topics of instruction, a cautious preceptor, every time he conducts his pupil over his first principles, and his successive elementary and other propositions, will show that fresh and more light is breaking in upon him, that his judgment as well as his knowledge, becomes the more amplified and confirmed; and that thus the student is seeing what evidence is, and in fact, seeing and feeling its authority and its beauty: duly and rightly taught, he feels himself emancipated from all bondage, but that of truth; and thus exulting in the new state of things, he travels on the road of demonstration easily, smoothly, and firmly, pursuing truth while he loves it more and more, and renders all truth subservient to all truth; the great object of our researches.

# TABLES

TO FACILITATE LOGARITHMIC AND TRIGONOMETRICAL COMPUTATION, FOR DIFFERENTIAL COEFFICIENTS, THE EXTRACTION OF ROOTS, &c.

## I. *Useful forms in Computation.*

The tables of sines, tangents, &c. besides their use in trigonometry, and in the solution of equations, are also very useful in finding the value of algebraic expressions, where extraction of roots would be otherwise required. Thus, if $a$ and $b$ be any two quantities, of which $a$ is the greater. Find $x$, $z$, &c., so, that

$\tan x = \sqrt{\frac{b}{a}}$, $\sin. z = \sqrt{\frac{b}{a}}$, $\sec. y = \frac{a}{b}$, $\tan. u = \frac{b}{a}$, and

$\sin. t = \frac{b}{a}$: then will

$\log. \sqrt{(a^2-b^2)} = \log. a + \log. \sin. y = \log. b + \log. \tan. y.$

$\log. \sqrt{(a^2-b^2)} = \frac{1}{2} [\log. (a+b) + \log. (a-b)].$

$\log. \sqrt{(a^2+b^2)} = \log. a + \log. \sec. u = \log. b + \log. \operatorname{cosec}. u.$

$\log. \sqrt{(a+b)} = \frac{1}{2}\log. a + \log. \sec. x = \frac{1}{2}\log. a + \frac{1}{2}\log. 2 + \log. \cos. \frac{1}{2}y.$

$\log. \sqrt{(a-b)} = \frac{1}{2}\log. a + \log. \cos. z = \frac{1}{2}\log. a + \frac{1}{2}\log. 2 + \log. \sin. \frac{1}{2}y.$

$\log. (a \pm b)^{\frac{m}{n}} = \frac{m}{n}[\log. a + \log. \cos t + \log. \tan. 45^\circ \pm \frac{1}{2}t].$

The first three of these formulæ will often be useful when two sides of a right-angled triangle are given, to find the third.

## II. *Useful Differential Coefficients (from Vega).*

| Coefficient. | Dec. Fraction. | Logarithm. | Coefficient. | Dec. Fraction. | Logarithm. |
|---|---|---|---|---|---|
| $\frac{1}{2}$ | 0·5000000000 | $\bar{1}$·6989700 | $\frac{1}{2.3}$ | 0.1666666667 | $\bar{1}$·2218487 |
| $\frac{1}{2.4}$ | 0·1250000000 | $\bar{1}$·0969100 | $\frac{1.3}{2.4.5}$ | 0·0750000000 | $\bar{2}$·8750613 |
| $\frac{1:3}{2.4.6}$ | 0·0625000000 | $\bar{2}$·7958800 | $\frac{1.3.5}{2.4.6.7}$ | 0·0446428571 | $\bar{2}$·6497519 |
| $\frac{1.3.5}{2.4.6.8}$ | 0·0390625000 | $\bar{2}$·5917600 | $\frac{1.3.5.7}{2.4.6.8.9}$ | 0·0303819444 | $\bar{2}$·4826156 |
| $\frac{1.3.5.7}{2.4......10}$ | 0·0273437500 | $\bar{2}$·4368653 | $\frac{1.3.5.7.9}{2.4.6.10.11}$ | 0·0223721591 | $\bar{2}$·3497078 |
| $\frac{1.3......9}{2.4......12}$ | 0·0205078125 | $\bar{2}$·3119212 | $\frac{1.3...9.11}{2.4...12.13}$ | 0·0173527644 | $\bar{2}$·2393687 |
| $\frac{1.3......11}{2.4......14}$ | 0·0161132812 | $\bar{2}$·2071840 | $\frac{1.3...11.13}{2.4...14.15}$ | 0·0139648437 | $\bar{2}$·1450360 |
| $\frac{1.3......13}{2.4......16}$ | 0·0130920410 | $\bar{2}$·1170073 | $\frac{1.3...13.15}{2.4...16.17}$ | 0·0115518009 | $\bar{2}$·0626497 |
| $\frac{1.3......15}{2.4......18}$ | 0·0109100342 | $\bar{2}$·0378260 | $\frac{1.3...15.17}{2.4..18.19}$ | 0·0097616095 | $\bar{3}$·9895214 |
| $\frac{1.3......17}{2.4......20}$ | 0·0092735291 | $\bar{3}$·9672450 | $\frac{1.3...17.19}{2.4...20.21}$ | 0·0083903358 | $\bar{3}$·9237794 |

| Coefficient. | Dec. Fraction. | Logarithm. | Coefficient. | Dec. Fraction. | Logarithm. |
|---|---|---|---|---|---|
| $\frac{1}{2}$ | 0·5000000000 | $\bar{1}$·6989700 | $\frac{1}{2.3}$ | 0·1666666667 | $\bar{1}$·2218487 |
| $\frac{1.3}{2.4}$ | 0·3750000000 | $\bar{1}$·5740313 | $\frac{1}{2.4.5}$ | 0·0250000000 | $\bar{2}$·3979400 |
| $\frac{1.3.5}{2.4.6}$ | 0·3125000000 | $\bar{1}$·4948500 | $\frac{1.3}{2.4.6.7}$ | 0·0089285714 | $\bar{3}$·9507819 |
| $\frac{1.3.5.7}{2.4.6.8}$ | 0·2734375000 | $\bar{1}$·4368653 | $\frac{1.3.5}{2.4.6.8.9}$ | 0·0043402778 | $\bar{3}$·6375174 |
| $\frac{1.3...... 9}{2.4......10}$ | 0·2460937500 | $\bar{1}$·3911005 | $\frac{1.3.5.7}{2.4...10.11}$ | 0·0024857954 | $\bar{3}$·3954754 |
| $\frac{1.3......11}{2.4......12}$ | 0·2255859375 | $\bar{1}$·3533119 | $\frac{1.3.5.7.9}{2.4...12.13}$ | 0·0015775251 | $\bar{3}$·1979763 |
| $\frac{1.3......13}{2.4......14}$ | 0·2094726562 | $\bar{1}$·3201273 | $\frac{1.3... 9.11}{2.4...14.15}$ | 0·0010742187 | $\bar{3}$·0310926 |
| $\frac{1.3......15}{2.4......16}$ | 0·1963806152 | $\bar{1}$·2930986 | $\frac{1.3...11.13}{2.4...16.17}$ | 0·0007701201 | $\bar{4}$·8865584 |
| $\frac{1.3......17}{2.4......18}$ | 0·1854705811 | $\bar{1}$·2682750 | $\frac{1.3...13.15}{2.4...18.19}$ | 0·0005742123 | $\bar{4}$·7590725 |
| $\frac{1.3......19}{2.4......20}$ | 0·1761970520 | $\bar{1}$·2459986 | $\frac{1.3...15.17}{2.4...20.21}$ | 0·0004415966 | $\bar{4}$·6450257 |

### III. *Rules for the Solution of Quadratics, by Tables of Sines and Tangents.*

1. If the equation be of the form $x^2 + px = q$:

Make tan. A $= \frac{2}{p}\sqrt{q}$; then will the two roots be, $x = +$ tan. $\frac{1}{2}$ A $\sqrt{q} \ldots\ldots x = -$ cot. $\frac{1}{2}$ A $\sqrt{q}$.

2. For quadratics of the form $x^2 - px = q$.

Make, as before, tan. A $= \frac{2}{p}\sqrt{q}$: then will

$x = -$ tan. $\frac{1}{2}$A $\sqrt{q} \ldots\ldots x = +$ cot. $\frac{1}{2}$A $\sqrt{q}$.

3. For quadratics of the form $x^2 + px + - q$.

Make sin. A $= \frac{2}{p}\sqrt{q}$: then will

$x = -$ tan. $\frac{1}{2}$A $\sqrt{q} \ldots\ldots x = -$ cot. $\frac{1}{2}$A $\sqrt{q}$.

4. For quadratics of the form $x^2 - px = - q$.

Make sin. A $= \frac{2}{p}\sqrt{q}$: then will

$x = +$ tan. $\frac{1}{2}$A $\sqrt{q} \ldots\ldots x = +$ cot. $\frac{1}{2}$A $\sqrt{q}$.

In the last two cases, if $\frac{2}{p}\sqrt{q}$ exceed unity, sin. A is imaginary, and consequently the values of $x$.

The logarithmic application of these formulæ is very simple.

Thus, in case 1st. Find A by making

$10 + \log. 2 + \frac{1}{2}\log. q - \log. p = \log. \tan. A.$

$$\text{Then log. } x. = \begin{cases} + \text{ log. tan. } \tfrac{1}{2} \text{ A} + \tfrac{1}{2}\text{log. } q - 10. \\ - (\text{log. cot. } \tfrac{1}{2}\text{A} + \tfrac{1}{2}\text{log. } q - 10). \end{cases}$$

*Note.* This method of solving quadratics is chiefly of use when the quantities $p$ and $q$ are large integers, or complex fractions.

## IV. *Rules for the Solution of Cubic Equations by Tables of Sines, Tangents, and Secants.*

1. For cubics of the form $x^3 + px \pm q = 0$.

Make $\tan.\ \text{B} = \frac{\frac{1}{3}p}{q} \,.\, 2\sqrt{\tfrac{1}{3}p} \ldots\ldots \tan.\ \text{A} = \sqrt[3]{\tan.\ \tfrac{1}{2}\text{B}}$.

Then $x = \mp \cot.\ 2\text{A} \,.\, 2\sqrt{p}$.

2. For cubics of the form $x^3 - px \pm q = 0$.

Make $\sin.\ \text{B} = \frac{\frac{1}{3}p}{q} \,.\, 2\sqrt{\tfrac{1}{3}p} \ldots\ldots \tan.\ \text{A} = \sqrt[3]{\tan.\ \tfrac{1}{2}\text{B}}$.

Then $x = \mp \operatorname{cosec}.\ 2\text{A} \,.\, 2\sqrt{\tfrac{1}{3}p}$.

Here, if the value of sin. B should exceed unity, B would be imaginary, and the equation would fall in what is called the *irreducible case* of cubics. In that case we must make $\operatorname{cosec}.\ 3\text{A} = \frac{\frac{1}{3}p}{q} \,.\, 2\sqrt{\tfrac{1}{3}p}$: and then the three roots would be

$$x = \pm \sin.\ \text{A} \,.\, 2\sqrt{\tfrac{1}{3}p}.$$
$$x = \pm \sin.\ (60^\circ - \text{A}) \,.\, 2\sqrt{\tfrac{1}{3}p}.$$
$$x = \pm \sin.\ (60^\circ + \text{A}) \,.\, 2\sqrt{\tfrac{1}{3}p}.$$

If the value of sin. B were 1, we should have $\text{B} = 90^\circ$, $\tan.\ \text{A} = 1$; therefore $\text{A} = 45^\circ$, and $x = \mp 2\sqrt{\tfrac{1}{3}p}$. But this would not be the only root. The second solution would give $\operatorname{cosec}.\ 3\text{A} = 1$: therefore $\text{A} = {}_{3}$ ; and then

$$x = \pm \sin.\ 30^\circ \,.\, 2\sqrt{\tfrac{1}{3}p} = \pm \quad \sqrt{\tfrac{1}{3}p}.$$
$$x \qquad \sin.\ 30^\circ \,.\, 2\sqrt{\tfrac{1}{3}p} = \pm \quad \sqrt{\tfrac{1}{3}p}.$$
$$x = \mp \sin.\ 90^\circ \,.\, 2\sqrt{\tfrac{1}{3}p} = \mp 2\sqrt{\tfrac{1}{3}p}.$$

Here it is obvious that the first two roots are equal, that their sum is equal to the third with a contrary

sign, and that this third is the one which is produced from the first solution.

In these solutions, the double signs in the value of $x$, relate to the double signs in the value of $q$.

### V. EXAMPLES OF ROOTS BY SINES AND TANGENTS.

*Example* 1. Find the roots of the equation $x^2 + \frac{7}{44}x = \frac{1695}{12716}$, by tables of sines and tangents.

Here $p = \frac{7}{44}$, $q = \frac{1695}{12716}$, and the equation agrees with the 1st form. Also tan. A $= \frac{88}{7}\sqrt{\frac{1695}{12716}}$ and $x =$ tan. $\frac{1}{2}$A $= \sqrt{\frac{1695}{12716}}$.

In Logarithms thus :

| | | |
|---|---|---|
| Log. 1695 = | 3·2291697 | |
| Arith. com. log. 12176 = | 5·8956495 | |
| sum + 10 = | 19·1248192 | |
| half sum = | 9·5624096 | |
| log. 88 = | 1·9444827 | |
| Arith. com. log. 7 = | 9·1549020 | |
| sum − 10 = log. tan. A = | 10·6617943 | = log. tan. 77° 42′ 31″¾; |
| log. tan. ½ A = | 9·9061115 | = log. tan. 38° 51′ 15″⅞; |
| log. $\sqrt{q}$, as above = | 9·5624096 | |
| sum − 20 = log. $x$ = − | 1·4685211 | = log. ·2941176. |

This value of $x$, viz. ·2941176, is nearly equal to $\frac{5}{17}$. To find whether that is the exact root, take the arithmetical complement of the last logarithm, viz. 0·5314379, and consider it as the logarithm of the denominator of a fraction whose numerator is unity:

thus is the fraction found to be $\frac{1}{3\cdot 4}$ exactly, and this is manifestly equal to $\frac{5}{17}$. As to the other root of the equation, it is equal to $-\frac{1695}{12716} \div \frac{5}{17} = -\frac{339}{748}$.

*Example* 2. Find the roots of the cubic equation $x^3 - \frac{403}{441}x + \frac{46}{147} = 0$, by a table of sines.

Here $p = \frac{403}{441}$, $q = \frac{46}{147}$, the second term is negative, and $4p^3 > 27q^2$: so that the example falls under the irreducible case.

Hence, $\sin. 3\text{A} = \frac{3 \times 46}{147} \times \frac{441}{403} \times \frac{1}{2\sqrt{\frac{403}{3\cdot 441}}} = \frac{14}{403} \cdot \frac{1}{\sqrt{\frac{1612}{1323}}}$.

The three values of $x$ therefore are

$$x = \sin. \text{A} \sqrt{\frac{1612}{1323}}.$$

$$x = \sin. (60^\circ - \text{A}) \sqrt{\frac{1612}{1323}}.$$

$$x = -\sin. (60^\circ + \text{A}) \sqrt{\frac{1612}{1323}}.$$

The logarithmic computation is subjoined.

| | | |
|---|---|---|
| Log. 1612 | = | 3·2073650 |
| Arith. com. log. 1323 | = | 6·8784402 |
| sum − 10...... | = | 0·0858052 |
| half sum | = | 0·0429026 const. log. |
| Arith. com. const. log. | = | 9·9570974 |
| log. 414 .. | = | 2 6170003 |
| Arith. com. log. 403 .. | = | 7·3946950 |
| log. sin. 3A .. | = | 9·9687927 = log. sin. 68° 32′ 18″¼. |

$$\text{Log. sin. A} = 9{\cdot}5891206$$
$$\text{const. log.} = \underline{0{\cdot}0429026}$$
$$\text{1. sum} - 10 = \text{log. } x = -\ \underline{\bar{1}{\cdot}6320232} = \text{log. } {\cdot}4285714 = \text{log. } \tfrac{3}{7}.$$
$$\text{Log. sin. } (60^\circ - \text{A}) = 9{\cdot}7810061$$
$$\text{const. log. } \ldots\ldots = \underline{0{\cdot}0429026}$$
$$\text{2. sum} - 10 = \text{log. } x = -\ \underline{\bar{1}{\cdot}8239087} = \text{log. } {\cdot}6666666 = \text{log. } \tfrac{2}{3}.$$
$$\text{Log. sin. } (60^\circ + \text{A}) = 9{\cdot}9966060$$
$$\text{const. log. } \ldots\ldots\ldots = \underline{0{\cdot}0429026}$$
$$\text{3. sum} - 10 = \text{log.} - x = 0{\cdot}0395086 = \text{log. } 1{\cdot}095238 = \text{log. } \tfrac{23}{21}.$$

So that the three roots are $\frac{3}{7}$, $\frac{2}{3}$, and $-\frac{23}{21}$; of which the first two are together equal to the third with its sign changed, as they ought to be.

### VI. SELECT EXAMPLES.

*For exercise, in finding the Roots of Equations by Horner's or equivalent method; all the Roots found.*

(1.) $x^3 - 12\,x = 15 \ldots\ldots\ldots\ldots\ x = \begin{cases} \ \ 3{\cdot}971963 \\ -1{\cdot}577032 \\ -2{\cdot}394930 \end{cases}$

(2.) $x^3 = 13x^2 + 49x - 45 = 0 \ldots\ x = \begin{cases} 5{\cdot} \\ 6{\cdot}64575 \\ 1{\cdot}35425 \end{cases}$

(3.) $x^3 - 6x = 2 \ldots\ldots\ldots\ldots\ldots\ x = \begin{cases} \ \ 2{\cdot}601676 \\ -2{\cdot}261806 \\ -\ \ {\cdot}339870 \end{cases}$

(4.) $x^3 - 27x = 36 \ldots\ldots\ldots\ldots\ x = \begin{cases} \ \ 5{\cdot}765722 \\ -4{\cdot}320684 \\ -1{\cdot}445038 \end{cases}$

(5.) $x^3 - 13x^2 + 38\,x + 16 = 0 \ldots\ x = \begin{cases} 8{\cdot} \\ 5{\cdot}3722823 \\ -{\cdot}3722823 \end{cases}$

(6.) $x^3 - 7x - 7 = 0$

$$x = 1{\cdot}6920214716300958696278148970020691 4$$
$$x = 1{\cdot}3568958678922094438943995190213005 8$$
$$-x = 3{\cdot}0489173395223053135222144070233697 2.$$

(7.) $x^3 - 7035x^2 + 15262754x - 10000730880 = 0.$
$x = 3456$, or $2345$, or $1234$.

(8.) Has the equation $x^4 - 4x^3 + 8x^2 - 16x + 20 = 0$, any real root ?

(9.) $x^4 - 6x^2 - 16x + 21 = 0 \ldots\ldots x = \begin{cases} 3\cdot \\ 1\cdot \\ 2 \pm \sqrt{-3} \end{cases}$

(10.) $x^4 - 19x^3 + 132x^2 - 302x + 200 = 0 \quad x = \begin{cases} 1\cdot 02804 \\ 4\cdot 00000 \\ 6\cdot 57653 \\ 7\cdot 39543 \end{cases}$

(11.) $x^4 - 4x^3 - 3x^2 - 4x + 1 = 0.$

Two real roots are ...................... $\begin{cases} 4\cdot 2912 \\ \cdot 2087 \end{cases}$

(12.) $x^4 - 36x^2 + 72x - 36 = 0 \ldots\ldots x = \begin{cases} 0\cdot 872984 \\ 1\cdot 267949 \\ 4\cdot 732051 \\ -6\cdot 872984 \end{cases}$

(13.) $x^4 + x^3 - 24x^2 + 43x = 21 \;..\; x = \begin{cases} 1\cdot \\ 3\cdot \\ 1\cdot 1400599 \\ -6\cdot 1400599 \end{cases}$

(14.) $x^4 - 27x^3 + 162x^2 + 356x = 1200 \;..\; x = \begin{cases} 2\cdot 05608 \\ -3\cdot 00000 \\ 13\cdot 15306 \\ 14\cdot 79086 \end{cases}$

(15.) $x^4 - 12x^2 + 12x - 3 = 0 \ldots x = \begin{cases} \cdot 606018 \\ -3\cdot 907378 \\ 2\cdot 858083 \\ \cdot 443277 \end{cases}$

(16.) $x^4 - 17x^2 - 20x - 6 = 0 \;..\; x = \begin{cases} 4.6457507 \\ \cdot 6457822 \\ 2 \pm \sqrt{2}. \end{cases}$

(17.) $x^4 + 24x^3 - 114x^2 - 24x + 1 = 0 \;..\; x = \begin{cases} 4\cdot 2360608 \\ \cdot 0356688 \\ -\cdot 236068 \\ -28\cdot 0356688 \end{cases}$

(18.) $x^4 - 112\cdot 3x^3 + 1243\cdot 53x - 2244\cdot 341x + 1112\cdot 111 = 0$
$x = 1\cdot 0$, or $10\cdot 1$, or $1\cdot 1$, or $100\cdot 1$.

(19) $x^5 - 3x^4 - 8x^3 + 24x^2 - 9x + 27 = 0$

$x = 3$, or $3\cdot$, or $-3\cdot$, or $\pm\sqrt{-1}$.

(20.) $\left\{\begin{array}{l} x^5 - 330x^4 + 35974x^3 - 1260072x^2 \\ \quad - 3733263x = 13676310. \end{array}\right\} x = \left\{\begin{array}{l} 111\cdot \\ 111\cdot \\ 111\cdot \\ -\frac{1}{2}(3 \pm \sqrt{-31}.) \end{array}\right.$

(21.) From M. Fourier, Analyse des Equations.

$$x^3 - {}^2x - 5 = 0$$

*Answer.* $x = 2\cdot094551481542326591482386540579302963857 6$; Fourier's operation, which includes five approximations, and occupies seven quarto pages, is carried to thirty-two figures. When properly worked by Horner's method, the solutions carried to forty figures, (if such an extensive approximation is necessary, as sometimes happens,) does not occupy any thing like a side of a sheet of paper. The other two roots are imaginary.

(22.) $x^6 - 6x^5 - 43x^4 + 95x^3 + 574x^2 + 149x - 182 = 0 \;\ldots\ldots\ldots\ldots\; x = \left\{\begin{array}{l} \cdot438448601 \\ 4\cdot56155012 \\ 8\cdot7958315 \\ -1\cdot2047625 \\ \text{Two impossible} \end{array}\right.$

(23.) $x^8 - 10x^7 - 10x^6 + 360x^5 + 2083x^4 - 3590x^3 + 5020x^2 - 5160x + 2016 = 0.$

*Answer.* $x = 1$, or 2, or 3, or 4, or 1, or $-2$, or $-6$, or 7.

(24.) $\left\{\begin{array}{l} \text{Given } x^7 - 5x^2y^4 + 1506 = 0 \;\ldots. \\ \text{and } y^5 - 3x^4y - 103 = 0 \;\ldots. \end{array}\right\}$ To find $x$ and $y$.

*Answer.* $x = 1\cdot996387$, $y - 3\cdot008239$.

VII. TABLE OF VERIFICATION FOR PLANE TRIANGLES.

| | | | |
|---|---|---|---|
| $a = 56\cdot925$, | $b = 45\cdot540$, | $C = 34\cdot154$, | |
| log. = 1·7553030... | 1·6583930...... | 1·5334543 | |
| A = 90° | B = 53° 7′ 48″, 4 | C = 36° 52′ 11″, 6 | |
| log.sin.B=9·9030900, | cos. B...9·7781512, | tang. B...0·1249389 | |
| | | log. $s$ | = 2·0357459, |
| $a = 57\cdot770$, | log. = 1·7617024, | log.$(s - a)$ | = 1·7059406, |
| $b = 71\cdot577$, | log. = 1·8647735, | log.$(s - b)$ | = 1·5682252, |
| $c = 87\cdot811$, | log. = 1·9435589, | log.$(s - c)$ | = 1·3173947, |
| A = 40° 56′ 00″, 00 | 9·8163609, | 9·8782186, | 9·9381423, |
| B = 54 . 16 . 8, 48, | 9·9094319, | 9·7663981, | 0·1430338, |
| C = 84 . 47 . 51, 52, | 9·9982073, | 8·9574805, | 11·0407268. |

VIII. TABLE OF VERIFICATION FOR RIGHT-ANGLED SPHERICAL TRIANGLES.

| Elements. | Log. Sines. | Log. Cosines. | Log. Tan. |
|---|---|---|---|
| $a$ = 71° 24′ 30″ | $\bar{1}$·9767235 | $\bar{1}$·5035475 + | 0·4731759 + |
| $b$ = 140° 52′ 40″ | $\bar{1}$·8000134 | $\bar{1}$·8897507 − | $\bar{1}$·9102626 − |
| $c$ = 114° 15′ 54″ | $\bar{1}$·9598303 | $\bar{1}$·6137969 − | 0·3460333 − |
| B = 138° 15′ 45″ | $\bar{1}$·8232909 | $\bar{1}$·8728568 − | $\bar{1}$·9504341 − |
| C = 105° 52′ 39″ | $\bar{1}$.9831068 | $\bar{1}$.4370867 − | 0·5460201 − |

The sign — that follows many of these logarithms is destined to indicate that the factor which it refers to is negative; which must not be confounded with the — which we place at the logarithms when we would indicate that it is to be subtracted, which happens in the case of a division. According as the number of negative factors of a formulæ is *even* or *odd*, the product has the sign + or —, a circumstance which we must note with care. Thus, for example, tan. $a$ has for $a$ an arc $a < a$, when that tangent is positive, and the supplement of that value when the tangent is negative.

IX. TABLE OF VERIFICATION FOR OBLIQUE-ANGLED SPHER. TRIANGLES.

| Elements. | Log. Sines. | Log. Cosines. | Log. Tan. |
|---|---|---|---|
| $a$ = 63° 39′ 57″ 8 | $\bar{1}$·9524165 | $\bar{1}$·6469937 | 0·3054227 |
| $b$ = 75° 0′ 51″ 3 | $\bar{1}$·9849727 | $\bar{1}$·4125922 | 0·5723798 |
| $c$ = 41° 9′ 46″ 0 | $\bar{1}$·8113582 | $\bar{1}$·8767042 | $\bar{1}$·2416540 |
| A = 66 . 57 . 3,6 | $\bar{1}$·9638682 | $\bar{1}$·5927520 | 0·3711162 |
| B = 97 . 20 . 31,6 | $\bar{1}$·9964244 | $\bar{1}$·1065091 − | 0·8892153 − |
| C = 42 . 30 . 55,0 | $\bar{1}$·8298097 | $\bar{1}$·8675247 | $\bar{1}$·9622849 |
| $\phi$ = 55 . 38 . 21,9 | $\bar{1}$·9467182 | $\bar{1}$·7515864 | 0·1651318 |
| $\phi'$ = − 14 . 28 . 35,9 | $\bar{1}$·3979144 − | $\bar{1}$·9859874 | $\bar{1}$·4119270 − |
| $\theta$ = 58 . 42 . 42,4 | $\bar{1}$·9317454 | $\bar{1}$·7154547 | 0·2162907 |
| $\theta'$ = − 16 . 11 . 47,4 | $\bar{1}$·4454990 − | $\bar{1}$·9824118 | $\bar{1}$·4630873 − |
| $\psi$ = 62 . 43 . 55,7 | $\bar{1}$·9488404 | $\bar{1}$·6610088 | 0·2878316 |

X. 2ND TABLE OF VERIFICATION FOR OBLIQUE-ANGLED SPHER. TRIANGLES.

| Elements. | Log. Sines. | Log. Cosines. | Log. Tan. |
|---|---|---|---|
| A = 121° 36′ 19″ 81 | $\bar{1}$·9302747 | $\bar{1}$·7193874 − | 0.2108873 − |
| B = 42° 15′ 13″ 66 | $\bar{1}$·8276379 | $\bar{1}$·8693336 | $\bar{1}$·9583043 |
| C = 34° 15′ 2″ 76 | $\bar{1}$·7503664 | $\bar{1}$·9172860 | $\bar{1}$·8330804 |
| $a$ = 76 . 35 . 36,00 | $\bar{1}$·9880008 | $\bar{1}$·3652279 | 0·6227729 |
| $b$ = 50 . 10 . 30,00 | $\bar{1}$·8853636 | $\bar{1}$·8064817 | 0·0788819 |
| $c$ = 40 . 0 . 10,00 | $\bar{1}$·8080926 | $\bar{1}$·8842363 | $\bar{1}$·9238563 |
| $\phi$ = − 32 . 8 . 50,00 | $\bar{1}$·7259905 − | $\bar{1}$·9277212 | $\bar{1}$·7982693 − |
| $\phi'$ = 72 . 9 . 0,00 | $\bar{1}$·9785741 | $\bar{1}$·4864674 | 0·4921067 |
| $\theta$ = − 43 . 51 . 16,20 | $\bar{1}$·8406263 − | $\bar{1}$·8579964 | $\bar{1}$·9826249 − |
| $\theta'$ = 78 . 6 . 19,00 | $\bar{1}$·9905733 | $\bar{1}$·3141076 | 0·6764657 |
| $\psi$ = 40 . 51 . 3,00 | $\bar{1}$·8156388 | $\bar{1}$·8787602 | $\bar{1}$·9368787 |

## XI. TRIGONOMETRICAL FORMULÆ;

***Either to assist the Memory in the Forms, or to select from them Examples of Forms for Investigation.***

---

### I. *Relative to a Single Arc or Angle a.*

1. $\sin.^2 a + \cos.^2 a = 1$
2. $\sin. a = \tan. a \cos a$
3. $\sin. a = \dfrac{\tan. a}{\sqrt{1 + \tan.^2 a}}$
4. $\cos. a = \dfrac{1}{\sqrt{1 + \tan.^2 a}}$
5. $\tan. a = \dfrac{\sin. a}{\cos. a}$
6. $\cot. a = \dfrac{1}{\tan. a} = \dfrac{\cos. a}{\sin. a}$
7. $\sin. a = 2 \sin. \frac{1}{2} a \cos. \frac{1}{2} a$
8. $\cos. a = 1 - 2 \sin.^2 \frac{1}{2} a$
9. $\cos. a = 2 \cos.^2 \frac{1}{2} a - 1$
10. $\tan. \frac{1}{2} a = \dfrac{\sin. a}{1 + \cos. a}$
11. $\cot. \frac{1}{2} a = \dfrac{\sin. a}{1 - \cos. a}$

12. $\tan.^2 \frac{1}{2} a = \frac{1 - \cos. a}{1 + \cos. a}$

13. $\sin. 2 a = 2 \sin. a \cos. a$

14. $\cos. 2 a = 2 \cos.^2 a - 1 = 1 - 2 \sin.^2 a$

II. *Relative to Two Arcs, a and b, of which a is supposed to be the greater.*

15. $\sin. (a + b) = \sin. a \cos. b + \sin. b \cos. a$

16. $\sin. (a - b) = \sin. a \cos. b - \sin. b \cos. a$

17. $\cos. (a + b) = \cos. a \cos. b - \sin. a \sin. b$

18. $\cos. (a - b) = \cos. a \cos. b + \sin. a \sin. b$

19. $\tan. (a + b) = \frac{\tan. a + \tan. b}{1 - \tan. a \tan. b}$

20. $\tan. (a - b) = \frac{\tan. a - \tan. b}{1 + \tan. a \tan. b}$

21. $\sin. a + \sin. b = 2 \sin. \frac{1}{2} (a + b) \cos. \frac{1}{2} (a - b)$

22. $\sin. a - \sin. b = 2 \sin. \frac{1}{2} (a - b) \cos. \frac{1}{2} (a + b)$

23. $\cos. a + \cos. b = 2 \cos. \frac{1}{2} (a + b) \cos. \frac{1}{2} (a - b)$

24. $\cos. b - \cos. a = 2 \sin. \frac{1}{2} (a + b) \sin. \frac{1}{2} (a - b)$

25. $\tan. a + \tan. b = \frac{\sin. (a + b)}{\cos. a \cos. b}$

26. $\tan. a - \tan. b = \frac{\sin. (a - b)}{\cos. a \cos. b}$

27. $\cot. a + \cot. b = \frac{\sin. (a + b)}{\sin. a \sin. b}$

28. $\cot. b - \cot. a = \frac{\sin. (a - b)}{\sin. a \sin. b}$

29. $\dfrac{\sin. a + \sin. b}{\sin. a - \sin. b} = \dfrac{\tan. \frac{1}{2}(a + b)}{\tan. \frac{1}{2}(a - b)}$

30. $\dfrac{\cos. b + \cos. a}{\cos. b - \cos. a} = \dfrac{\cot. \frac{1}{2}(a + b)}{\tan. \frac{1}{2}(a - b)}$

31. $\dfrac{\tan. a + \tan. b}{\tan. a - \tan. b} = \dfrac{\cot. b + \cot. a}{\cot. b - \cot. a} = \dfrac{\sin. (a + b)}{\sin. (a - b)}$

32. $\dfrac{\cot. b - \tan. a}{\cot. b + \tan. a} = \dfrac{\cot. a - \tan. b}{\cot. a + \tan. b} = \dfrac{\cos. (a + b)}{\cos. (a - b)}$

33. $\sin.^2 a - \sin.^2 b = \sin. (a + b) \sin. (a - b)$

34. $\cos.^2 a - \sin.^2 b = \cos. (a + b) \cos. (a - b)$

35. $1 \pm \sin. a = 2 \sin.^2 (45° \pm \frac{1}{2} a)$

36. $\dfrac{1 \pm \sin. a}{1 \mp \sin. a} = \tan.^2 (45° \pm \frac{1}{2} a)$

37. $\dfrac{1 \pm \sin. a}{\cos. a} = \tan. (45° \pm \frac{1}{2} a)$

38. $\dfrac{1 - \sin. a}{1 - \cos. a} = \dfrac{\sin.^2 (45° - \frac{1}{2} a)}{\sin.^2 \frac{1}{2} a}$

39. $\dfrac{1 + \sin. b}{1 + \cos. a} = \dfrac{\sin.^2 (45° + \frac{1}{2} b)}{\cos.^2 \frac{1}{2} a}$

40. $\dfrac{1 + \tan. b}{1 - \tan. b} = \tan. (45° + b)$

41. $\dfrac{1 - \tan. b}{1 + \tan. b} = \tan. (45° - b)$

42. $\sin. a \cos. b = \frac{1}{2} \sin. (a + b) + \frac{1}{2} \sin. (a - b)$

43. $\cos. a \sin. b = \frac{1}{2} \sin. (a + b) - \frac{1}{2} \sin. (a - b)$

44. $\sin. a \sin. b = \frac{1}{2} \cos. (a - b) - \frac{1}{2} \cos. (a + b)$

45. $\cos. a \cos. b = \frac{1}{2} \cos. (a + b) + \frac{1}{2} \cos. (a - b)$

## XII. *Differences of Trigonometrical Lines.*

46. $\Delta \sin. B = 2 \sin. \frac{1}{2} \Delta B \cos. (B + \frac{1}{2} \Delta B).$

47. $- \Delta \cos. B = 2 \sin. \frac{1}{2} \Delta B \sin. (B + \frac{1}{2} \Delta B).$

48. $\Delta \tan. B = \dfrac{\sin. \Delta B}{\cos. B \cos. (B + \Delta B)}.$

49. $- \Delta \cot. B = \dfrac{\sin. \Delta B}{\sin. B \sin. (B + \Delta B)}.$

50. $\left.\begin{array}{r} \Delta (\sin.^2 B) = \\ - \Delta (\cos.^2 B) = \end{array}\right\} \sin. \Delta B \sin. (2B + \Delta B).$

51. $\Delta (\tan.^2 B) = \dfrac{\sin. \Delta B \sin. (2B + \Delta B)}{\cos.^2 B \cos.^2 (B + \Delta B)}$

52. $- \Delta (\cot.^2 B) = \dfrac{\sin. \Delta B \sin. (2B + \Delta B)}{\sin.^2 B \sin.^2 (B + \Delta B)}.$

## XIII. *Differentials of Trigonometrical Lines.*

53. $d \sin. B = d B \cos. B.$

54. $- d \cos. B = d B \sin. B.$

55. $d \tan. B = \dfrac{dB}{\cos.^2 B}.$

56. $- d \cot. B = \dfrac{d B}{\sin.^2 B}.$

57. $\left.\begin{array}{r} d (\sin.^2 B) = \\ - d (\cos.^2 B) = \end{array}\right\} 2d B \sin. B \cos. B.$

58. $d (\tan.^2 B) = \dfrac{2d B \tan. B}{\cos.^2 B}.$

59. $- d (\cot.^2 B) = \dfrac{2d B \cot. B}{\sin.^2 B}.$

## XIV. BUCK'S TRIGONOMETRICAL MULTIPLICATION AND DIVISION TABLES.

### Multiplied into

| | sin. | cosec. | tan. | cot. | sec. | cos. |
|---|---|---|---|---|---|---|
| sin. | $\text{sin.}^2$ | rad. | $\frac{\text{sin.}}{\text{cot.}}$ | cos. | tan. | $\frac{\text{sin.}}{\text{sec.}}$ |
| cosec. | rad. | $\text{cosec.}^2$ | sec. | $\frac{\text{cosec.}}{\text{tan.}}$ | $\frac{\text{cosec.}}{\text{cos.}}$ | cot. |
| tan. | $\frac{\text{sin.}}{\text{cot.}}$ | sec. | $\text{tan.}^2$ | rad. | $\frac{\text{tan.}}{\text{cos.}}$ | sin. |
| cot. | cos. | $\frac{\text{cosec.}}{\text{tan.}}$ | rad. | $\text{cot.}^2$ | cosec. | $\frac{\text{cot.}}{\text{sec.}}$ |
| sec. | tan. | $\frac{\text{cosec.}}{\text{cos.}}$ | $\frac{\text{tan.}}{\text{cos.}}$ | cosec. | $\text{sec.}^2$ | rad. |
| cos. | $\frac{\text{sin.}}{\text{sec.}}$ | cot. | sin. | $\frac{\text{cot.}}{\text{sec.}}$ | rad. | $\text{cos.}^2$ |

### Divided by

| | sin. | cosec. | tan. | cot. | sec. | cos. |
|---|---|---|---|---|---|---|
| sin. | 1 | $\text{sin.}^2$ | cos. | sin. tan. | sin. cos. | tan. |
| cosec. | $\text{cosec.}^2$ | 1 | cosec. cot. | sec. | cot. | cosec. sec. |
| tan. | sec. | tan. sin. | 1 | $\text{tan.}^2$ | sin. | tan. sec. |
| cot. | $\frac{\text{cosec.}}{\text{tan.}}$ | cos. | $\text{cot.}^2$ | 1 | $\frac{\text{cos.}}{\text{tan.}}$ | cosec. |
| sec. | sec. cosec. or $\frac{\text{cosec.}}{\text{cos.}}$ | tan. | cosec. | sec. tan. | 1 | $\text{sec.}^2$ |
| cos. | $\frac{\text{sin.}}{\text{sec.}}$ or sin. cos. | cot. | $\frac{\text{cot.}}{\text{sec.}}$ | sin. | $\text{cos.}^2$ | 1 |
| unity | cosec. | sin. | cot. | tan. | cos. | sec. |

XV. *Solution of Spherical Triangles, by Gauss's Formulæ.*

(I.) $\cos. \frac{1}{2}(a+b) . \cos. \frac{1}{2} C = \cos. \frac{1}{2}(A+B) . \sin. \frac{1}{2} c$

(II.) $\cos. \frac{1}{2}(a \backsim b) . \sin. \frac{1}{2} C = \sin. \frac{1}{2}(A+B) . \sin. \frac{1}{2} c$

(III.) $\sin. \frac{1}{2}(a+b) . \cos. \frac{1}{2} C = \cos. \frac{1}{2}(A \backsim B) . \cos. \frac{1}{2} c$

(IV.) $\sin. \frac{1}{2}(a \backsim b) . \sin. \frac{1}{2} C = \sin. \frac{1}{2}(A \backsim B) . \cos. \frac{1}{2} c$

*The Four Fundamental Formulæ of Spherical Trigonometry.*

| | |
|---|---|
| (1.) Three sides and an angle, A.B.C. $a$. | $\cos. A = \cos. a \sin. B. \sin. C + \cos. B. \cos. C.$ |
| (2.) Three angles and a side, $a.b.c.$ A. | $\cos. a = \cos. A \sin. b \sin. c - \cos. b \cos. c.$ |
| (3.) Four contiguous parts, A.$b$. C.$a$. | $\cot. A \sin. C = \cot. a \sin. b + \cos. C \cos. b.$ |
| (4.) Four opposite parts, A.$a$. B.$b$. | $\frac{\sin. A}{\sin. a} = \frac{\sin. B.}{\sin. b.}$ |

XVI. *Regular Polygons. The side Unity.*

| Sides. | Names. | Areas. | Log. Areas. |
|---|---|---|---|
| 3 | Trigon, or equilat. △ | 0.4330127 | $\bar{1}$.636501 |
| 4 | Tetragon, or square | 1.0000000 | 0.000000 |
| 5 | Pentagon | 1.7204774 | 0.235649 |
| 6 | Hexagon | 2.5980762 | 0.414652 |
| 7 | Heptagon | 3.6339124 | 0.560374 |
| 8 | Octagon | 4.8284271 | 0.683806 |
| 9 | Nonagon | 6.1818242 | 0.791117 |
| 10 | Decagon | 7.6942088 | 0.886164 |
| 11 | Undecagon | 9.3656405 | 0.971537 |
| 12 | Dodecagon | 11.1961524 | 1.049069 |

*General Construction of Polygons. Corrected from Malton.*

Bisect AB the diameter of the circle in C, through which, perpendicular to AB, draw indefinitely HD. Take AI to AB as AG is to AGB, and draw GID. Then will D be the point whence a right line being drawn through the given point I, will, as is required, divide the diameter and concave circumference in the same ratio. Trigon, CD = 1·73205; Pentagon, 1·74478; Hexagon, 1·73205; Heptagon, 1·71903; Octagon, 1·707106; Nonagon, 1·69654; Decagon, 1·68728; Undecagon, 1·679165; Dodecagon, 1·67202.

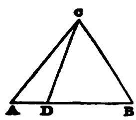
# APPENDIX.

## PROBLEMS RELATIVE TO THE DIVISION OF FIELDS OR OTHER SURFACES.

### PROBLEM I.

To divide a triangle into two parts having a given ratio, $m : n$.

1. By a line drawn from one angle of a triangle.

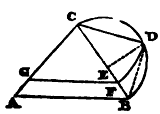


Make AD : AB :: $m : m + n$; draw C. So shall ADC, BDC, be the parts required.

Here, evidently, AD $= \frac{m}{m+n}$ AB, DB $= \frac{n}{m+n}$ AB.

2. By a line parallel to one of the sides of the triangle.

Let ABC be the given triangle, to be divided into two parts, in the ratio of $m$ to $n$, by a line parallel to the base AB. Make CE to EB as $m$ to $n$; erect ED perpendicularly to CB, till it meet the semicircle described on CB, as a diameter in D. Make

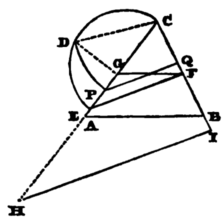


$CF = CD$ : and draw through F, GF $\|$ AB. So shall GF divide the triangle ABC in the given ratio.

For, $CE : CB = \frac{CD^2}{CE} :: CD^2 (= CF^2) : CB^2$. But, $CE : EB :: m : n$, or $CE : CB :: m : m + n$, by the construction; therefore, $CF^2 : CB^2 :: m : m + n$. And since $\triangle\ CGF : \triangle\ CAB :: CF^2 : CB^2$; it follows that $CGF : CAB :: m : m + n$, as required.

*Computation.*—Since $CB^2 : CF^2 :: m + n : m$, therefore, $(m + n)\ CF^2 = m \ . \ CB^2$; whence $CF \sqrt{(m + n)} = CB \sqrt{m}$, or $CF = CB \sqrt{\frac{m}{m + n}}$. In like manner, $CG = CA \sqrt{\frac{m}{m + n}}$.

3. By a line parallel to a given line.




Let HI be the line parallel to which a line is to be drawn, so as to divide the triangle ABC in the ratio of $m$ to $n$.

By case second draw GF parallel to AB, so as to divide ABC in the given ratio. Through F draw FE parallel to HI. On CE as a diameter describe a semicircle; draw GD perp. to AC, to cut the semicircle in D. Make $CP = CD$; through P, parallel to EF, draw PQ, the line required.

The demonstration of this follows at once from case 2; because it is only to divide FCE, by a line parallel to FE, into two triangles having the ratio of FCE to FCG, that is, of CE to CG.

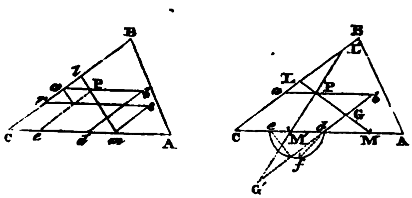
*Computation.*—CG and CF being computed, as in case 1, the distances CH, CI being given, and CP being to CQ as CH to CI : the triangles CGF, CPQ, also having a common vertical angle, are to each other as CG . CF to CQ . CP. These products therefore are equal; and since the factors of the former are known, the latter product is known. We have hence given the ratio of the two lines CP ( $= x$) to CQ ( $= y$) as CH to CI ; say, as $p$ to $q$ ; and their product $=$ CF . CG, say $= ab$ : to find $x$ and $y$.

Here we find $x = \sqrt{\frac{abq}{q}}$, $y = \sqrt{\frac{abq}{p}}$. That is,

$$\text{CP} = \sqrt{\frac{\text{CF . CG . CH}}{\text{CI}}}\text{; CQ} = \sqrt{\frac{\text{CF . CG . CI}}{\text{CH}}}.$$

N.B—If the line of division were to be perpendicular to one of the sides, as to CA, the construction would be similar ; CP would be a geometrical mean between CA and $\frac{m}{m+n}$C$b$, $b$ being the foot of the perpendicular from B upon AC.

4. By a line drawn through a given point P.




By any of the former cases draw $lm$ (fig. 1) to divide the triangle ABC, in the given ratio of $m$ to $n$ ; bisect C$l$ in $r$, and through $r$ and $m$ let pass the sides of the

rhomboid $Crsm$. Make $Ca = Pe$, which is given because the point P is given in position; make $Cd$ a fourth proportional to $Ca$, $Cr$, $Cm$; that is, make $Ca : Cr :: Cm : Cd$; and let $a$ and $d$ be two angles of the rhomboid $Cabd$, figs. 1 and 2. $Pe$, in figure 2, being drawn parallel to $a$C, describe on $ed$ as a diameter the semicircle $efd$, on which set off $ef = Ce = a$P; then set off $d$M or or$d$M′ on CA equal to $df$, and through P and M, P and M′ draw the lines LM, L′M′, either of which will divide the triangle in the given ratio. The construction is given in two figures merely to avoid complexness in the diagrams.

The limitations are obvious from the construction; for the point L must fall between B and C, and the point M between A and C; $a$P must also be less than P$b$, otherwise $ef$ cannot be applied to the semicircle on $ed$.

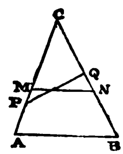
*Demonstration.*—Because $Cr = \frac{1}{2}\, Cl$ the rhomboid $Crsm$ = triangle $Clm$, and because $Ca : Cr :: Cm : Cd$, we have $Ca \cdot Cd = Cm \cdot Cr$, therefore rhomboid $Cabd$ = rhomboid $Crsm$ = triangle $Clm$. By reason of the parallels CB, $bd$, and CA, $ab$, the triangles $a$LP, $d$GM, $b$GP, are similar, and are to each other as the squares of their homologous sides $a$P, $d$M, $b$P; now $ed^2 = ef^2 + df^2$, by construction; and $ed = Pb$, $ef = a$P, $df = d$M; therefore $Pb^2 = aP^2 + dM^2$, or, the triangle P$b$G taken away from the rhomboid, is equal to the sum of the triangles $a$PL, $d$MG, added to the part C$a$PG$d$; consequently CLM = C$abd$, as required. By a like process it may be shown that $a$L′P, $d$ G′M′, P$b$G′, are similar,

and $aL'P + dG'M' = PbG'$; whence $PbdM' = aL'P$, and $CL'M' = Cabd$, as required.

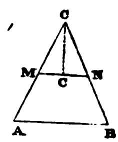
*Computation.*—$Cl$, $Cm$, being known, as well as $Ca$, $aP$, or $Ce$, $eP$, $Cr = \frac{1}{2}Cl$, is known; and hence $Cd$ may be found by the proportion $Ca : Cr :: Cm : Cd$. Then $Cd - Ce = ed$, and $\sqrt{ed^2 - ef^2} = \sqrt{ed^2 - aP^2} = df = dM = dM'$. Thus CM is determined. Then we have $\frac{Cl.Cm}{CM} = CL$.

N.B. When the point is in one of the sides, as at M; then make $CL . CM \ (m + n) = CA . CB . m$, or $CL : CA : m . CB : (m + n) CM$, and the thing is done.

5. By the shortest line possible.




Draw any line PQ dividing the triangle in the given ratio, and so that the summit of the triangle CPQ shall be C the *most acute* of the three angles of the triangle. Make CM = CN, a geometrical mean proportional between CP and CQ; so shall MN be the shortest line possible dividing the triangle in the given ratio. The computation is evident.




*Demonstration.*—Suppose MN to be the shortest line cutting off the given triangle CMN, and CG $\perp$ MN, $MN = MG + GN = CG . \cot. M + CG . \cot. N = CG (\cot. M + \cot. N)$. But, $\cot. M + \cot. N = \frac{\cos. M}{\sin. N} + \frac{\cos. N}{\sin. N} =$

$\frac{\sin.(M+N)}{\sin. M.\sin. N}$. And (equ. XVIII, ch. iii.) sin. M . sin. N $= \frac{1}{2}\cos.(M-N) - \frac{1}{2}\cos.(M+N) = \frac{1}{2}\cos.(M-N) + \frac{1}{2}\cos. C$. Therefore, MN = CG. $\frac{\sin.(M+N)}{\frac{1}{2}\cos.(M-N)+\frac{1}{2}\cos. C}$; which expression is a minimum when its denominator is a maximum; that is, when cos. (M — N) is the greatest possible, which is manifestly when M — N = 0, or M = N, or when the triangle CMN is isosceles. That the isosceles triangle must have the most *acute* angle for its summit, is evident from the consideration, that since 2 △ CMN = CG, MN, MN varies inversely as CG; and consequently MN is shortest when CG is longest, that is, when the angle C is the most acute.

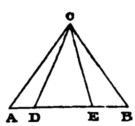
N.B. A very simple and elegant demonstration to this case is given in Simpson's Geometry: vide the book on Max. and Min. See also another demonstration at case 2, prob. 6, below.

PROBLEM II.

To divide a triangle into three parts, having the ratio of the quantities *m*, *n*, *p*.

1. By lines drawn from one angle of the triangle to the opposite side.

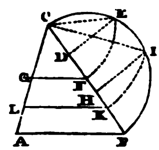
Divide the side AB, opposite the angle C from whence the lines are to proceed, in the given ratio at D, E; join CD, CE; and ACD, DCE, ECB, are the three triangles required. The demonstration is manifest; as is also the computation.



If it be wished that the lines of division be the shortest the nature of the case will admit of, let them be drawn from the most obtuse angle to the opposite or *longest* side.

2. By lines parallel to one of the sides of the triangle.

Make CD : DH : HB :: $m : n : p$. Erect DE, HI, perpendicularly to CB, till they meet the semicircle described on the diameter CB, in E and I. Make CF = CE, and CK = CI. Draw GF through F, and LK through K, parallel to AB; so shall the lines GF and LK divide the triangle ABC as required.



The demonstration and computation will be similar to those in the second case of prob. 1.

3. By lines drawn from a given point on one of the sides.

*Fig.* 1.

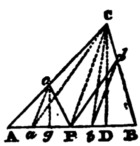
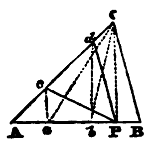


*Fig* 2.



Let P (fig. 1) be the given point, $a$ and $b$ the points which divide the side AB in the given ratio of $m, n, p$; the point P falling between $a$ and $b$. Join PC, parallel to which draw $ac$, $bd$, to meet the sides AC, BC, in the points $c$ and $d$; join P$c$, P$d$, so shall the lines $c$P, P$d$, divide the triangle in the given ratio.

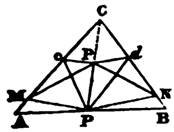
In fig. 2, where P falls nearer one of the extremities of

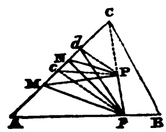
AB than both $a$ and $b$, the construction is essentially the same ; the sole difference in the result is, that the points $e$ and $d$ both fall on *one* side AC of the triangle.

*Demonstration.*—The lines C$a$, C$b$, divide the triangle into the given ratio, by case 1. But by reason of the parallel lines $ac$ PC, $bd$, $\triangle\ ac\ \mathrm{C} = \triangle\ ac\mathrm{P}$, and $\triangle\ bd\mathrm{C} = bd\mathrm{P}$. Therefore, in fig. 1, $\mathrm{A}ac + ac\mathrm{P} = \mathrm{A}ac + ac\mathrm{C}$, that is, $\mathrm{A}c\mathrm{P} = \mathrm{A}a\mathrm{C}$ ; and $\mathrm{B}bd + bd\mathrm{P} = \mathrm{B}bd + bd\mathrm{C}$, that is, $\mathrm{B}d\mathrm{P} = \mathrm{B}b\mathrm{C}$. Consequently, the remainder $\mathrm{C}c\mathrm{P}d = \mathrm{C}ab$. In fig. 2, $\mathrm{A}c\mathrm{P} = \mathrm{A}a\mathrm{C}$, and $\mathrm{A}d\mathrm{P} = \mathrm{AC}b$; therefore $c\mathrm{P}d = a\mathrm{CP}$; and $\mathrm{ACB} - \mathrm{A}d\mathrm{P} = \mathrm{ACB}, \mathrm{AC}b$, that is, $\mathrm{CBP}d = \mathrm{CB}b$.

*Computation.*—The perpendiculars $cg$, CD being demitted, $\triangle\ \mathrm{A}c\mathrm{P} : \triangle\ \mathrm{ACB} :: m : m + n + p :: \mathrm{AP} \,.\, cg : \mathrm{AB} \,.\, \mathrm{CD}$. Therefore $(m + n + p)\,\mathrm{AP} \,.\, \mathrm{C}g = m \,.\, \mathrm{AB} \,.\, \mathrm{CD}$, and $cg = \dfrac{m \,.\, \mathrm{AB} \,.\, \mathrm{CD}}{(m + n + p)\,\mathrm{AP}}$. The line $cg$ being thus known, we soon find A$c$; for $\mathrm{CD} : \mathrm{AC} :: cg : \mathrm{A}c = \dfrac{\mathrm{AC} \,.\, cg}{\mathrm{CD}} = \dfrac{m \,.\, \mathrm{AB} \,.\, \mathrm{AC}}{(m + n + p)\,\mathrm{AP}}$. Indeed this expression may be deduced more simply; for, since $\mathrm{ACB} : \mathrm{A}c\mathrm{P} :: \mathrm{AC} \,.\, \mathrm{AB} : \mathrm{A}c \,.\, \mathrm{AP} :: m + n + p : m$, we have $(m + n + p)\,\mathrm{A}c \,.\, \mathrm{AP} = m \,.\, \mathrm{AB} \,.\, \mathrm{AC}$, and $\mathrm{A}c = \dfrac{m \,.\, \mathrm{AB} \,.\, \mathrm{AC}}{(m + n + p)\,\mathrm{AP}}$. By a like process is obtained, in fig. 1, $\mathrm{B}d = \dfrac{p \,.\, \mathrm{AB} \,.\, \mathrm{BC}}{(m + n + p)\,\mathrm{PB}}$; and in fig. 2, $\mathrm{A}d = \dfrac{(m + n)\,\mathrm{AB} \,.\, \mathrm{AC}}{(m + n + p)\,\mathrm{AP}}$.

**4. By lines drawn from a given point P *within* the triangle.**




*Const.*—Through P and C draw the line C P $p$, and let the triangle be divided into the given ratio by lines $pc$, $pd$, drawn from $p$ to intersect AC, BC, or either of them; according to the method described in case 3 of this problem. Through P draw P$c$, P$d$, and respectively parallel to them, from $p$ draw the lines $p$M, $p$N : join PM, PN ; so shall these lines with P$p$, divide the triangle in the given ratio.

*Demon.*—The triangles $c$PM, $c$P$p$, are manifestly equal, as are also $d$PN, $d$P$p$; therefore CPM = C$pc$, and CPN = C$pd$; whence also, in fig. 1, CNPM = C$dpc$ and in fig. 2, CB $p$ PN = CB$pd$.

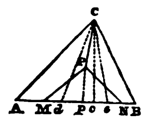
*Comput.*—Since CP . CN = C$p$ . C$d$, we have CN = $\frac{Cp \cdot Cd}{CP}$.

In like manner CM = $\frac{Cp \cdot Cc}{CP}$.

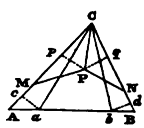
*Remark.*—It will generally be best to contrive that the *smallest* share of the triangle shall be laid off nearest the vertex C of the triangle, in order to ensure the possibility of the construction. Even this precaution, however, may sometimes fail of ensuring the construction

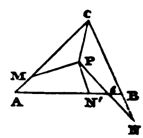
by the method above given. When this happens, proceed thus:

By case 1, draw the lines C*d*, C*e*, from the vertex C to the opposite side AB, to divide the triangle in the given ratio. Upon AB set off any where MN, so that MN : AB :: P*p* (the perp. from P on AB) : C*p*, the altitude of the triangle. If MP and PN are together to be the least possible, then set off $\frac{1}{2}$ MN on each side of the point *p*; so will the triangle MPN be isosceles, and its perimeter (with the given base and area) a minimum.




5. By lines, *one* of which is drawn *from* a given angle to a given point, which is also the point of concourse of the other two lines.




*Const.*—By case 1st draw the lines C*a*, C*b*, dividing the triangle in the given ratio, and so that the smaller portions shall lie nearest the angles A and B (unless the conditions of the division require it to be otherwise). From P and *a* demit upon AC the perpendiculars P*p*, *ac*; and from P and *b*, on BC, the perpendiculars P*q*, *bd*. Make CM : CA :: *ac* : P*p*, and CN : CB :: *bd* : P*q*. Draw PM, PN, which, with CP, will divide the triangle as required.

When the perpendicular from *b* or from *a*, upon BC or

AC, is longer than the corresponding perpendicular from P, the point N or M will fall further from C than B or A does. Suppose it to be N; then make N′*e* : *e*B :: N*e* : *e*P, and draw PN′ for the line of division.

The demonstration of all this is too obvious to need tracing here.

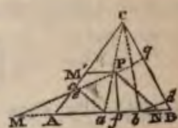
*Comput.*—The perp. $ca = \mathrm{A}a$ . sin. A ; and

$$\mathrm{CM} = \frac{\mathrm{CA} \,.\, ac}{\mathrm{P}p}.$$

$$bd = \mathrm{B}b \text{ . sin. B ; and } \mathrm{CN} = \frac{\mathrm{CB} \,.\, bd}{\mathrm{P}q}.$$

6. By lines, one of which falls from the given point of concourse of all three, upon a given side, in a given angle.

Suppose the given angle to be a right angle, and P*f* the given perpendicular: which will simplify the operation, though the principles of construction will be the same.




*Const.*—Let C*a* C*b* divide the triangle in the given ratio. Make *f*N : CB :: *bd* : P*f*, and *f*M : CA :: *ac* : P*f*; and draw PN, PM, thus forming two triangles P*f*N, P*f*M, equal to C*b*B, C*a*A respectively. If N fall between *f* and B, and M between A and *f*, this construction manifestly effects the division. But if one of the points, suppose M, falls beyond the corresponding point A, the line PM intersecting AC in *e*. Then make M′*e* : *e*A :: *e*M : *e*P, and

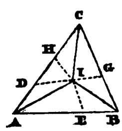
draw PM′ : so shall P$f$, PM′, PN, divide the triangle as required.

*Comput.*—Here $ca$ and $bd$ are found as in case 5; and hence $fN = \frac{CB \cdot bd}{Pf}$; and $fM = \frac{CA \cdot ac}{Pf}$. Then PM $= \sqrt{(Mf^2 + fP^2)}$, and $\frac{Pf}{PM}$ = sin. M. Also 180° − (M + A) = M$e$A. Then sin. M$e$A : sin. M : sin. A ∝ MA (= M$f$ − A$f$) : A$e$ : M$e$. Again P$e$ = PM − M$e$; and lastly $M'e = \frac{Ae \cdot eM}{eP}$.

Here also the demonstration is manifest.

7. By lines drawn from the angles to meet in a determinate point.

*Construc.*—On one of the sides, as AC, set off AD, so that AD : AC :: $m : m + n + p$. And on any other, as AB, set off BE, so that BE : BC :: $n : m + n + p$. Through D draw DG parallel to AB; and through E, EH parallel to BC; to their point of intersection I draw the lines AI, BI, CI, which will divide the triangle ABC into the portions required.



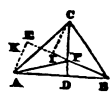

*Demon.*—Any triangle whose base is AB, and whose vertex falls in DG parallel to it, will manifestly be to ABC, as AD to AC, or as $m$ to $m + n + p$; so also, any triangle whose base is BC, and whose vertex falls in EH parallel to it, will be to ABC, as BE to BA, that is, as $n$ to $m + n + p$.

Thus we have AIB : ABC :: $m : m + n + p$.
and . . BIC : ACB :: $n : m + n + p$,
therefore . AIB : BIC :: $m : n$.

And the first two proportions give, by composition, AIB + BIC : ACB :: $m + n : m + n + p$; and by division, ACB − (AIB + BIC) : ACB :: $m + n + p - (m + n) : m + n + p$, or AIC : ACB :: $p : m + n + p$, consequently AIB : BIC : AIC $\propto$ $m : n : p$.

*Comput.*—BE = GI = $\frac{n \cdot AB}{m + n + p}$; BG = $\frac{m \cdot BC}{m + n + p}$; angle BGI = 2 right angles − B. Hence, in the triangle BGI, there are known two sides and the included angle, to find the third side BI.

*Remark.*—When $m = n = p$, the construction becomes simpler. Thus, from the vertex draw CD to bisect AB; and from B draw BE in like manner to the middle of AC; the point of intersection I of the lines CD, BE, will be the point sought.



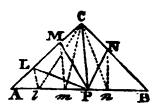

For, on BE and BE produced, let fall from the angles C and A, the perpendiculars CI, AK; then the triangles CEI, AEK, are equal in all respects, because AE = CE, KAE = ICE, and the angles at E are equal. Hence AK = CI. But these are the perpendicular altitudes of the triangles BPC, BPA, which have the common base BP. Consequently those two triangles are equal in area. In a similar manner it may be proved, that

APC = APB or CPB. Therefore these three triangles are equal to each other, and the lines PA, PB, PC, trisect the △ ABC.

PROBLEM III.

To divide a triangle into four parts, having the proportion of the quantities *m*, *n*, *p*, *q*.

This, like the former problems, might be divided into several cases, the consideration of all which would draw us to a very great length, and which is in great measure unnecessary, because the method will in general be suggested immediately on contemplating the method of proceeding in the analogous case of the preceding problem. We shall therefore only take one case, namely, that in which the lines of division must all be drawn from a given point in one of the sides.




Let P be the given point in the side AB.

Let the points *l*, *m*, *n*, divide the base AB in the given proportion; so will the lines C*l*, C*m*, C*n*, divide the surface of the triangle in the same proportion. Join CP, and parallel to it draw, from *l*, *m*, *n*, the lines *l*L, *m*M, *n*N, to cut the other two sides of the triangle in L, M, N. Draw PL, PM, PN, which will divide the triangle as required.

The demonstration is too obvious to need tracing throughout; for the triangles L*l*P, L*l*C, having the same base L*l*, and lying between the same two parallels L*l*, C*p*, are equal; to each of these adding the triangle AL*l*,

there results ALP = AC*l*. And in like manner the truth of the whole construction may be shown.

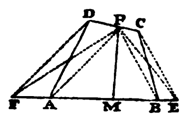
The computation may be conducted after the manner of that in case 3, prob. 2.

PROBLEM IV.

To divide a quadrilateral into two parts having a given ratio, *m* : *n*.

1. By a line drawn from any point in the perimeter of the figure.

*Construc.*—From P draw lines PA, PB, to the opposite angles A, B. Through D draw DF parallel to PA, to meet BA produced in F; and through C draw CE parallel to PB to meet AB produced in E. Divide FE in M, in the given ratio of *m* to *n*; join P, M; so shall the line PM divide the quadrilateral as required.




*Demon.*—That the triangle FPE is equal to the quadrangle ABCD, may be shown by the same process as is used to demonstrate the construction of prop. 36, Geometry; of which, in fact, this is only a modification. And the line PM evidently divides EFE in the given ratio. But FPM = ADPM, and EPM=BPCM; therefore PM divides the quadrangle also in the given ratio.

*Remark* 1. If the line PM cut either of the sides, AD, BC, then its position must be changed by a process similar to that described in the 5th and 6th cases of the last problem.

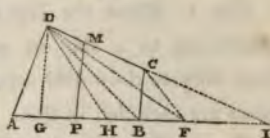
*Remark* 2. The quadrilateral may be divided into three, four, or more parts, by a similar method, being subject however to the restriction mentioned in the preceding remark.

*Remark* 3. The same method may obviously be used when the given point P is in one of the angles of the figure.

*Comput.*—Suppose I to be the point of intersection of the sides DC and AB, produced; and let the part of the quadrilateral laid off towards I, be to the other, as $n$ to $m$. Then we have $IM = \frac{n\,(ID\,.\,IA - IB\,.\,IC)}{(m+n)\,IP.}$ As to the distances DI, AI, (since the angles at A and D, and consequently that at I, are known), they are easily found from the proportionality of the sides of triangles to the sines of their opposite angles.

2dly. By a line drawn parallel to a given line.

*Construc.*—Produce DC, AB, till they meet, as at I. Join DB, parallel to which draw CF. Divide AF in the given ratio in H. Through D draw DG parallel to the given line. Make IP a mean proportional between IH, IG; through P draw PM parallel to GD: so shall PM divide the quadrilateral ABCD as required.




*Demon.*—It is evident, from the transformation of figures, so often resorted to in these problems, that the triangle ADF = quadrilateral ABCD (th. 36 Geom.):

and that DH divides the triangle ADF in the given ratio, is evident from prob. 1, case I. We have only then to demonstrate that the triangle IHD is equal to the triangle LPM, for in that case HDF will manifestly be equal to BCMP. Now, by construction, IH . IP : : IP : IG : : (by the parallels) IM : ID ; whence, by making the products of the means and extremes equal, we have ID . IH =IP . IM ; but when the products of the sides about the equal angles of two triangles having a common angle are equal, those triangles are equal ; therefore $\Delta$ IHD = $\Delta$ IPM. Q. E. D.

*Comput.*—In the triangles ADI, ADG, are given all the angles, and the side AD ; whence AI, AG, DI, and IC, = DI − DC, become known. In the triangle IFC, all the angles and the side IC are known ; whence IF becomes known, as well as FH, since AH : HF : : $m$ : $n$. Lastly, IP = $\sqrt{(\text{IH . IG})}$, and IG : ID : : IP : IM.

*Cor.* 1. When the line of division PM is to be perpendicular to a side, or parallel to a given side ; we have only to draw DG accordingly : so that those two cases are included in this.

*Cor.* 2. When the line PM is to be the shortest possible, it must cut off an isosceles triangle towards the acutest angle ; and in that case IG must evidently be equal to ID.

3dly. By a line drawn through a given point.

The method will be the same as that to case 4th, prob. 1, and therefore need not be repeated here.

*Scholium.*—If a quadrilateral were to be divided into

four parts in a given proportion, $m$, $n$, $p$, $q$: we must first divide it into two parts having the ratio of $m + n$ to $p + q$; and then each of the quadrangles so formed into their respective ratios, of $m$ to $n$, and $p$ to $q$.

### PROBLEM V.

To divide a pentagon into two parts having a given ratio, from a given point in one of the sides.

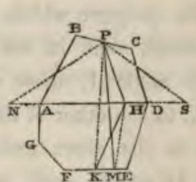
Reduce the pentagon to a triangle by prob. 37, geometry, and divide this triangle in the given ratio by case 1, prob. 1.

### PROBLEM VI.

To divide *any* polygon into two parts having a given ratio.

1st. From a given point in the perimeter of the polygon.

*Construc.*—Join any two opposite angles, A, D, of the polygon, by the line AD. Reduce the part ABCD into an equivalent triangle NPS, whose vertex shall be the given



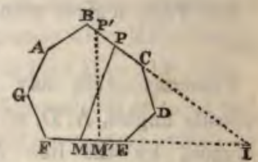

point P, and base AD produced: an operation which may be performed at once, if the portion ABCD be quadrangular; or by several operations (as from 8 sides to 6, from 6 to 4, &c.) if the sides be more than four. Divide the triangle NPS into two parts having the given ratio, by the line PA. In like manner, reduce ABEFGA into an equivalent triangle having H for its vertex, and

FE produced for its base; and divide this triangle into the given ratio by a line from H, as HK. The compound line PHK will manifestly divide the whole polygon into two parts having the given ratio. To reduce this to a right line, join PK, and through H draw HM parallel to it; join PM; so will the right line PM divide the polygon as required, provided M fall between F and E. If it do not, the reduction may be completed by the process described in cases 5th and 6th, prob. 2nd.

All this is too evident to need demonstration.

*Remark.*—There is a *direct* method of solving this problem, without subdividing the figure; but as it requires the computation of the area, it is not given here.

2dly.—By the shortest line possible.




*Construc.*—From any point P′, in one of those two sides of the polygon which, when produced, meet in the most acute angle I, draw a line P′M′, to the other of those sides (EF), dividing the polygon in the given ratio. Find the points P and M, so that IP or IM shall be a mean proportional between IP′, IM′; then will PM be the line of division required.

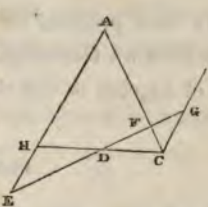
The demonstration of this is the same as has been already given, at case 5, prob. 1. Those, however, who wish for a proof, independent of the arithmetic of sines, will not be displeased to have the additional demonstration below.

The *shortest* line which, with two other lines given in

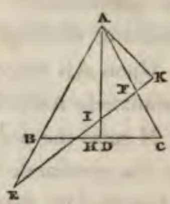
position, includes a given area, will make equal angles with those two lines, or with the segments of them it cuts off from an isosceles triangle.

Let the two triangles ABC, AEF, having the common angle A, be equal in surface, and let the former triangle be isosceles, or have AB = AC; then is BC shorter than EF.

First, the oblique base EF cannot pass through D, the middle point of BC, as in the annexed figure. For, drawing CG parallel to AB, to meet EF produced in G. Then the two triangles DBE, DCG are identical, or mutually equal in all respects. Consequently the triangle DCF is less than DBE, and therefore ABC less than AEF.




EF must therefore cut BC in some point H between B and D, and cutting the perp. AD in some point I above D, as in the 2d figure. Upon EF (produced if necessary) demit the perp. AK. Then, in the right-angled Δ AIK, the perp. AK is less than the hypothenuse AI, and therefore *à fortiori* less than the other perp. AD. But of equal triangles, that which has the greatest perpendicular, has the least base. Therefore the base BC is less than the base EF. Q. E. D.



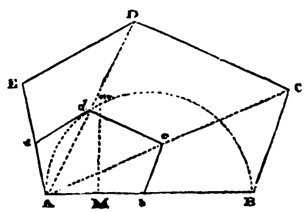
This series of problems might have been extended much further; but the preceding will furnish a sufficient variety, to suggest to the student the best method to be adopted in almost any other case that may occur. The following practical examples are subjoined by way of exercise.

We here present, for the guidance of the student, from *Hirsch's Geometry*, a few examples of the application of Algebra to this class of problems.

### PROBLEM VII.

To divide a given polygon into two parts according to a given proportion, and in such a manner, that one of the parts may be similar to the whole figure.

Divide one of the sides of the polygon, say AB, according to the given proportion: let the point of division be M. Upon AB describe the semicircle A*m*B, and




from M raise the perpendicular upon AB that is M*m*, to meet the semicircle in *m*. Make A*b* = A*m*, and upon AB describe, by drawing the lines respectively parallel to BC, CD, &c. the figure A*bcde*, which is evidently similar to ABCDE; and it divides the figure in the manner required.

*Demon.* ABCDE : *abcde* :: $AB^2 : ab^2$;
:: $AB^2 : Am^2$ :: AB : AM;

consequently:

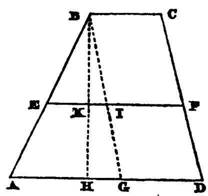
ABCDE*e* : A*bcde* : A*bcde* :: AB—AM : AM,

or, BCDE*edcb* : A*bcde* :: MB : AM.

PROBLEM VIII.

From a given quadrilateral with two parallel sides, to cut off, by a line parallel to these sides, a part consisting of a given area.

*Solut.*—Let ABCD be the trapezium, with two parallel sides AD, BC, from which trapezium, by a line EF, parallel to these two sides, it is required to cut off a part BCFE, whose area $= q$.




1. Draw the perpendicular BH, and BG parallel to CD, and let AD $= a$, BC $= b$, and the altitude BH $= h$. If we knew how to determine the point K, in which the lines BH, EF intersect each other, we could then draw the line of division. Let ∴ BK $= x$, and EF $= y$.

2. Since EI is parallel to AG, therefore

$$AG : EI = BG : BI = BH : BK$$

or $$a - b : y - b = h : x$$

consequently $$(a - b)\, x = (y - b)\, h$$

3. Trapez. $$BCFE = \tfrac{1}{2}\,(y + b)\, x = q$$

consequently $(y + b)\, x = 2q.$

4. Therefore the two equations 2 and 3, when solved, give—

$$y = \sqrt{\left[\frac{2q\,(a-b)}{h} + b^2\right]}$$

$$x = \frac{h}{a-b}\left[-b + \sqrt{\left(\frac{2q\,(a-b)}{h} + b^2\right)}\right]$$

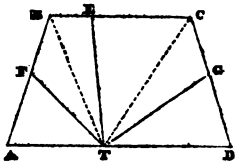
*Exam.*—Let $a = 76$, $b = 36$, $h = 23$, and $\therefore$ the area of the trapezium $= 1288$; it is required to cut off from it a part containing 560; what is the length of the line of section EF, and its distance from BC?

Ans. EF $= 56{\cdot}954$, and BK $= 12{\cdot}048$.

PROBLEM IX.

To divide a trapezium with two parallel sides, in a given proportion, from a given point in one of its sides.

*Solut.*—Let ABCD be the trapezium, which from the point T is to be divided by a line TE in such a way that the section ABET is to the whole trapezium, as $n : m$. Let AD $= a$, BC $= b$, AB $= c$, CD $= d$, AT $= f$.




1. Put BE $= x$, then, because the altitudes are equal,

Trapez. ABCD : trapez. : ABET :: $a + b : f + x$;
But
Trapez. ABCD : trapez. : ABET :: $m : n$
consequently $a + b : f + x :: m : n$

$$\therefore\ x = \frac{n\,(a+b)}{m} - f.$$

2. If $x$ in the course of the operation be found to be

negative, this indicates, that the point E is not situated in BC, but in AB. In this case, let TF be the line of section, and AF $= y$. Draw BT; then

Trapez. ABCD : $\triangle$ ATB :: $a + b : f$

and, $\triangle$ATB : $\triangle$ ATF :: $c : y$;

consequently,

Trapez. ABCD : $\triangle$ ATF :: $c(a + b) : fy$.

But, Trapez. ABCD : $\triangle$ ATF :: $m : n$.

consequently, $c(a+b) : fy :: m : n$.

$$\therefore y = \frac{nc\,(a + b)}{mf}.$$

3. If in 1 $x$ be found greater than $b$, this indicates that the line of section must fall in CD. Let $\therefore$ TG be the line of section, and DG $= z$. Draw CT; then

Trapez. ABCD : $\triangle$ CTD :: $a + b : a - f$

and $\triangle$ CDT : $\triangle$ GTD :: $d : z$

consequently,

Trapez. ABCD : $\triangle$ GTD :: $d(a + b) : (a - f)z$.

But by hypothesis, Trapez. ABCD : ABCGT :: $m : n$

and $\therefore$ Trapez. ABCD : $\triangledown$ GTD :: $m : m - n$.

We have $\therefore d(a + b) : (a - f)z = m : m - n$

and $$z = \frac{(m - n)(a + b)d}{m(a - f)}$$

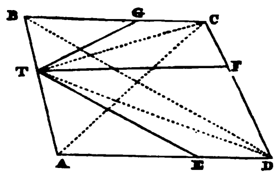
*Exam.*—Let $a = 112$, $b = 80$, $c = 45$, $d = 40$, $f = 30$. If it is required to cut off the third part from the trapezium, assume BE $= x = 34$, and draw TE. If the tenth part is to be cut off, assume AF $= y = 28{\cdot}8$, and draw TF; but if $\frac{5}{8}$ths of the trapezium are to be

cut off, assume DG $= x = 35 \cdot 112$, or as near as may be, and draw TG.

PROBLEM X.

To divide a trapezium having two parallel sides, in a given proportion, from a given point not in the parallel sides.

*Solut.*—Let ABCD be the trapezium; AD, BC, the parallel sides, and T the point from which the line of section is drawn.




First calculate the triangles ATD, CTD, BTC, with reference to the trapezium; then from the magnitude of these triangles, and from the magnitude of the part to be cut off, we may easily judge whether the point E in the line of section TE, falls in AD, CD, or BC. Let ∴ AD $= a$, BC $= b$, TA $= c$, TB $= d$, and the area of the trapezium $=$ A.

1. If we draw the lines AC, BD, then we find, by similar methods to those already employed,

$$\text{ATD} = \frac{ac}{(a+b)(c+d)} \cdot \text{ A};$$

$$\triangle \text{ BTC} = \frac{bd}{(a+b)(c+d)} \cdot \text{ A}$$

$$\triangle \text{ CTD} = \text{Trapez. ABCD} - \triangle \text{ ATD} - \triangle \text{ BTC}$$

$$= \frac{ad+bc}{(a+b)(c+d)} \cdot \text{ A}.$$

2. If it is required to cut off a given part from the trapezium, it is only necessary to divide one of these

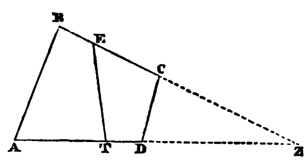
triangles, from its vertex T, in a given proportion, which is effected by dividing its base in this proportion. The following example will elucidate this.

*Exam.*—Let $a = 120$, $b = 98$, $c = 46$, $d = 31$; also $\triangle$ ATD = 0·3288 . A, $\triangle$ BTC = 0·1810 . A, $\triangle$ CTD = 0·4902 . A. If it is required to cut off the fourth part of the trapezium, or $\frac{1}{4}$ A = 0·25 . A, we must then divide AD in E, so that AD : AE = 3288 : 2500, and then, if we draw TE, TAE is the fourth part. If it is required to cut off $\frac{2}{3}$rds of the trapezium, or $\frac{2}{3}$ A = 0·6666 . A, we must in this case add a $\triangle$ DTF = 0·3378 to ATD, and consequently divide DC in F in such a manner that DC : DF = 4902 : 3378; then ATFD will be the part required. If we wish to cut off the $\frac{8}{9}$th part, or $\frac{8}{9}$ A = 0·8888 . A, we must, because $\triangle$ ATD + $\triangle$ DTC = 0·8190 . A, add a $\triangle$ CTG = 0·0698 . A to the quadrilateral ATCD, and consequently divide BC in G, so that BC : CG = 1810 : 698; then ATGCD will be the part required.

## PROBLEM XI.

From a given point to divide any trapezium in a given proportion.

*Solut.*—Let ABCD be the given trapezium, and T the point from which the line of division TE is so drawn that trapez. ABCD : trapez. DCET = $m$ : $n$.



1. Produce the sides BC, AD till they meet in Z. Since the trapez. ABCD is given, the lines AZ, BZ, CZ, DZ, may also be determined. Let $\therefore$ AZ $= a$, BZ $= b$, DZ $= c$, CZ $= d$. Since the point T is also given, let ZT $= f$. In order now to determine the point E, we put ZE $= x$.

2. The rules for the areas of plane triangles.

$$\triangle \text{BZA} : \triangle \text{CZD} :: ab : cd$$

$\therefore \triangle$ BZA $- \triangle$ CZD : $\triangle$ CZD :: $ab - cd : cd$

or trapez. ABCD : $\triangle$ CZD :: $ab - cd : cd$

3. In like manner, $\triangle$ EZT : $\triangle$ CZD :: $fx : cd$.

$\therefore \triangle$ EZT $- \triangle$ CZD : $\triangle$CZD :: $fx - cd : cd$.

or trapez. DCET : $\triangle$ CZD :: $fx - cd : cd$.

4. From 2 and 3 we obtain

Trapez. ABCD : trapez. DCET :: $ab - cd : fx - cd$

But trapez. ABCD : trapez. DCET :: $m : n$

consequently $ab - cd : fx - cd :: m : n$.

and
$$x = \frac{n\,(ab - cd)}{mf} + \frac{cd}{f}$$

*Exam.*—Let $a = 200$, $b = 178$, $c = 112$, $d = 120$, $f = 140$, and it is required to cut off $\frac{3}{8}$ths of the trapezium. Assume ZE $= 155\frac{5}{14}$, and draw TE; then DCET is the part required.

### *Miscellaneous Exercises.*

*Ex.* 1. A triangular field, whose sides are 20, 18, and 16 chains, is to have a piece of four acres in content fenced off from it, by a right line drawn from the most obtuse angle to the opposite side. Required the length

of the dividing line, and its distance from either extremity of the line on which it falls?

*Ex.* 2. The three sides of a triangle are 5, 12, and 13. If two-thirds of this triangle be cut off by a line drawn parallel to the longest side, it is required to find the length of the dividing line, and the distance of its two extremities from those of the longest side.

*Ex.* 3. It is required to find the length and position of the shortest possible line, which shall divide, into two equal parts, a triangle whose sides are 25, 24, and 7 respectively.

*Ex.* 4. The sides of a triangle are 6, 8, and 10: it is required to cut off nine-sixteenths of it, by a line that shall pass through the centre of its inscribed circle.

*Ex.* 5. Two sides of a triangle, which include an angle of 70°, are 14 and 17 respectively. It is required to divide it into three equal parts, by lines drawn parallel to its longest side.

*Ex.* 6. The base of a triangle is 112·65, the vertical angle 57° 57′, and the difference of the sides about that angle is 8. It is to be divided into three equal parts, by lines drawn from the angles to meet in a point within the triangle. The lengths of those lines are required.

*Ex.* 7. The legs of a right-angled triangle are 28 and 45. Required the lengths of lines drawn from the middle of the hypothenuse, to divide it into four equal parts.

*Ex.* 8. The length and breadth of a rectangle are 15 and 9. It is proposed to cut off one-fifth of it, by a line which shall be drawn from a point on the longest side at the distance of 4 from a corner.

*Ex.* 9. A regular hexagon, each of whose sides is 12, is to be divided into four equal parts, by two equal lines; both passing through the centre of the figure. What is the length of those lines when a minimum?

*Ex.* 10. The three sides of a triangle are 5, 6, and 7. How may it be divided into four equal parts, by two lines which shall cut each other perpendicularly?

The student will find that some of these examples will admit of two answers.

I conclude this Appendix with answers to Euler's questions relative to the circle, given in the 22d chapter of his Infinitesimal Analysis.

### PROBLEM I.

To find the arc of a circle which is equal to its cosine.

*Solut.*—The arc which is equal to its cosine, radius 1, is $42^\circ\ 20'\ 47''\ 14'''$, the arc itself $= 0{\cdot}7390847$.

### PROBLEM II.

To find the sector of the circle ACB, which is divided by the chord AB into two equal parts, so that the triangle ACB shall be equal to the segment AEB.

*Solut.*—The arc shall be $54^\circ\ 18'\ 6''\ 52'''\ 44''''$, its sine AF = BF = ·8121029, the chord AB = 1·6242058, its cosine CF = ·5335143.

PROBLEM III.

To draw in a quadrant ACB a sine DE, which shall divide the area into two equal parts.

*Solut.*—We shall have the arc AE $= 66° \; 10' \; 23'' \; 37'''$, and the arc BE $= 23° \; 49' \; 36'' \; 23'''$, the part CD of the radius is $= \cdot 4039718$, AD $= 0 \cdot 5960281$, and the sine DE $= 0 \cdot 9147711$. The same means which will divide the quadrant into two equal parts, will divide the whole circle into eight equal parts.

PROBLEM IV.

Having given the semicircle AEDB, to draw from the point A a chord AD which will divide the surface into two equal parts.

*Solut.*—We shall have by the first prob. $u = s - 90° = 42° \; 20' \; 47'' \; 14'''$, $s$, or ACD $= 130° \; 20' \; 47'' \; 14'''$, the angle BCD $= 47° \; 39' \; 12'' \; 46'''$; the chord AD will be $= 1 \cdot 8295422$.

PROBLEM V.

From a point A of the circumference of a circle, to draw two chords AB, AC, which shall divide the area of the circle into three equal parts.

*Solut.*—The arc AEB $=$ AEC $= 149° \; 16' \; 27''$, the arc BC $= 61° \; 27' \; 6''$, and the chord AB $=$ AC $= 1 \cdot 9285340$.

PROBLEM VI.

To cut from the semicircle AEB the arc AE, so that

in dropping its sine ED, the arc AE shall be equal to the sum of the right lines, that is AD + DE.

*Solut.*—The arc sought is AE $=$ 138° 11′ 53″ 0‴; and the lines are DE $=$ 0·6665578, and AD $=$ 1·7454535. If DE were $=$ ·6666666 or $\frac{2}{3}$, then AD + DE would have been $= 1 + \frac{2}{3} + \sqrt{\frac{5}{9}}$.

### PROBLEM VII.

To find a sector ACD which shall be equal to the half of the triangle formed by the radius AC, the tangent AE, and the secant CE.

*Solut.*—The arc AD $=$ 66° 46′ 54″ 14,‴, and the tangent AB $=$ 2·3311220.

### PROBLEM VIII.

Having given a quadrant ACB, to find the arc AE equal to its chord AE prolonged until it meets the radius CB continued to F.

*Solut.*—The arc $s$ $=$ AE $=$ 84° 53′ 38″ 51‴, the arc BE $=$ 5° 6′ 21″ 9‴.

Euler obtained these results by the method of trial and error, and tables of circular arcs, and logarithmic sines and tangents: some of the same are done similarly by *Hirsch*.

THE END.

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